Frontiers of Network Science
Fall 2021

Class 16: Degree Correlations
(Chapter 7 in Textbook)

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based on slides by
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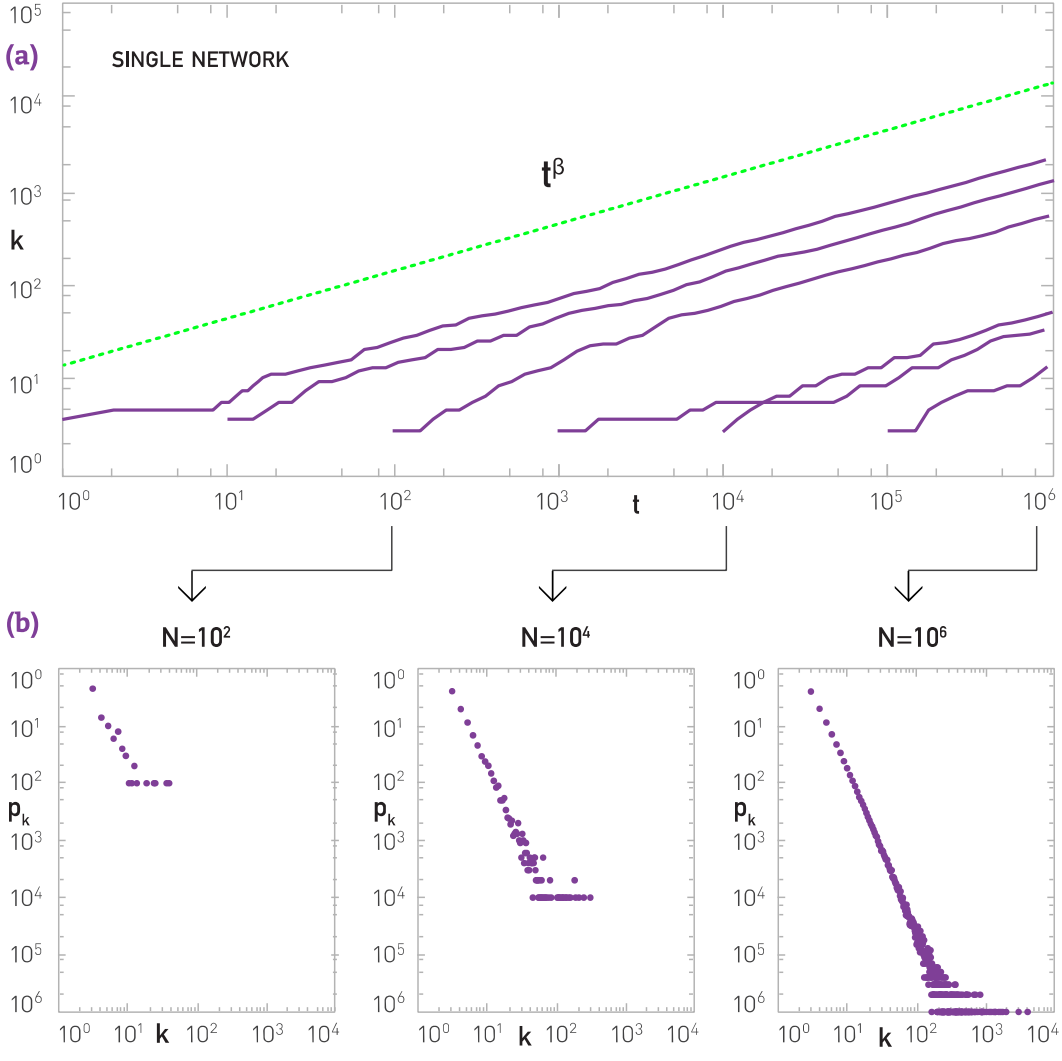
Section 5.3



- The degree of each node increases following a power-law with the same dynamical exponent β =1/2 (Figure 5.6a). Hence all nodes follow the same dynamical law.

- The growth in the degrees is sublinear (i.e. $\beta < 1$). This is a consequence of the growing nature of the Barabási-Albert model: Each new node has more nodes to link to than the previous node. Hence, with time the existing nodes compete for links with an increasing pool of other nodes.

- The earlier node i was added, the higher is its degree $k_i(t)$. Hence, hubs are large because they arrived earlier, a phenomenon called *first-mover advantage* in marketing and business.

- The rate at which the node i acquires new links is given by the derivative of (5.7)

$$\frac{dk_i(t)}{dt} = \frac{m}{2}\frac{1}{\sqrt{t_i t}}, \tag{5.8}$$

indicating that in each time frame older nodes acquire more links (as they have smaller t_i). Furthermore the rate at which a node acquires links decreases with time as $t^{-1/2}$. Hence, fewer and fewer links go to a node.

Absence of growth and preferential attachment

MODEL A

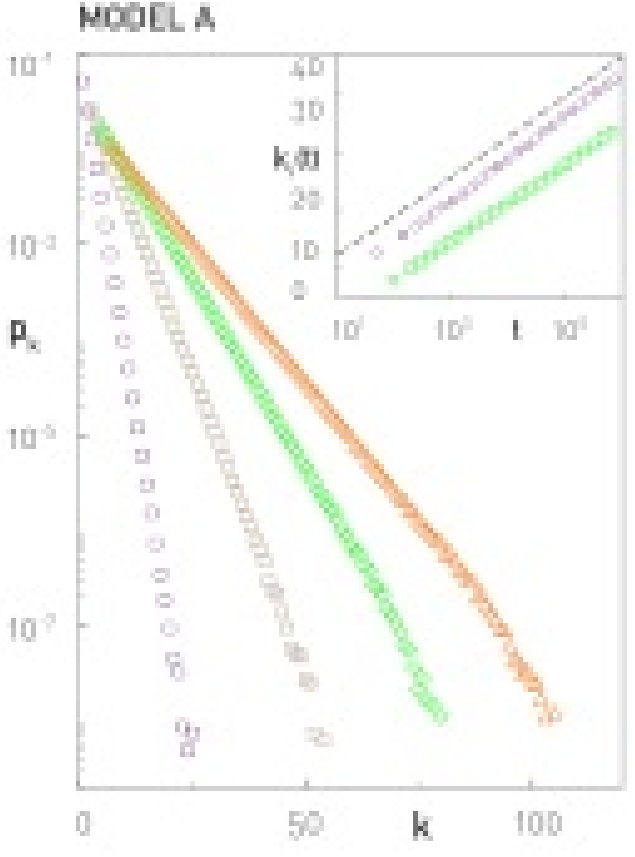
growth ~~preferential attachment~~

$\Pi(k_i)$: uniform

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) = \frac{m}{m_0 + t - 1}$$

$$k_i(t) = m \ln\left(\frac{m_0 + t - 1}{m + t_i - 1}\right) + m$$

$$P(k) = \frac{e}{m}\exp\left(-\frac{k}{m}\right) \sim e^{-k}$$

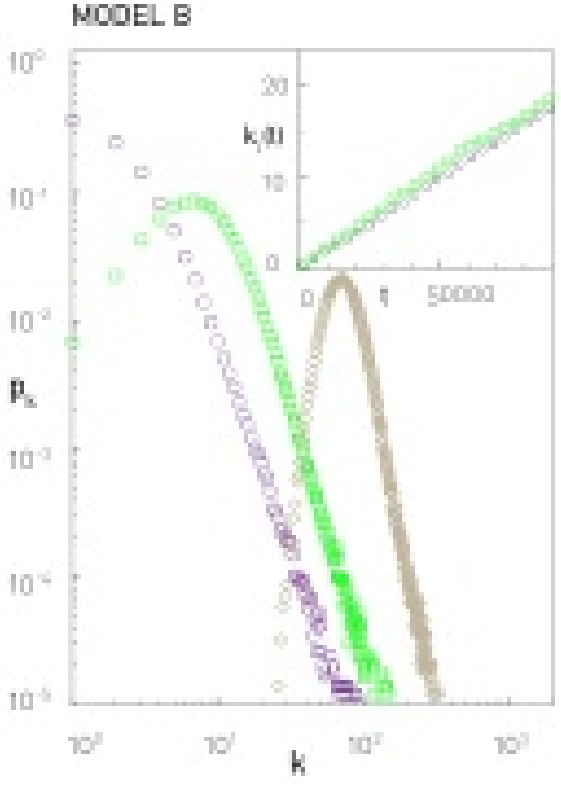


MODEL B

~~growth~~ preferential attachment

$$\frac{\partial k_i}{\partial t} = A\Pi(k_i) + \frac{1}{N} = \frac{N}{N-1}\frac{k_i}{2t} + \frac{1}{N}$$

$$k_i(t) = \frac{2(N-1)}{N(N-2)}t + Ct^{\frac{N}{2(N-1)}} \sim \frac{2}{N}t$$



p_k : power law (initially) →

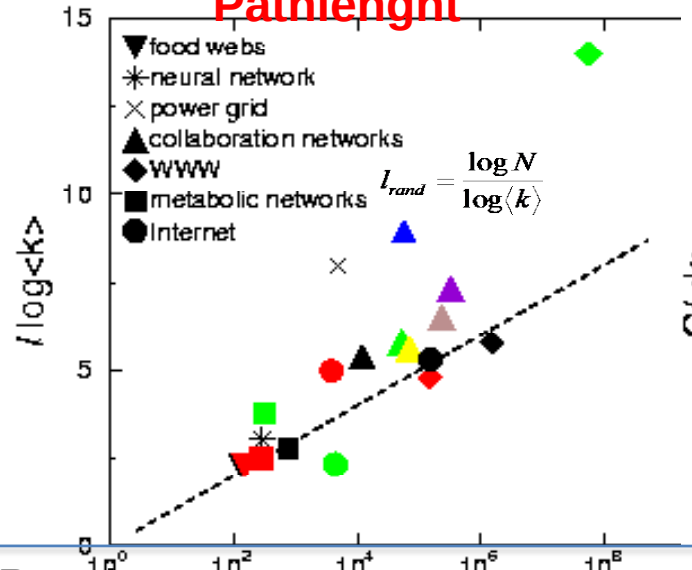
→ Gaussian → Fully Connected

Do we need both growth and preferential attachment?

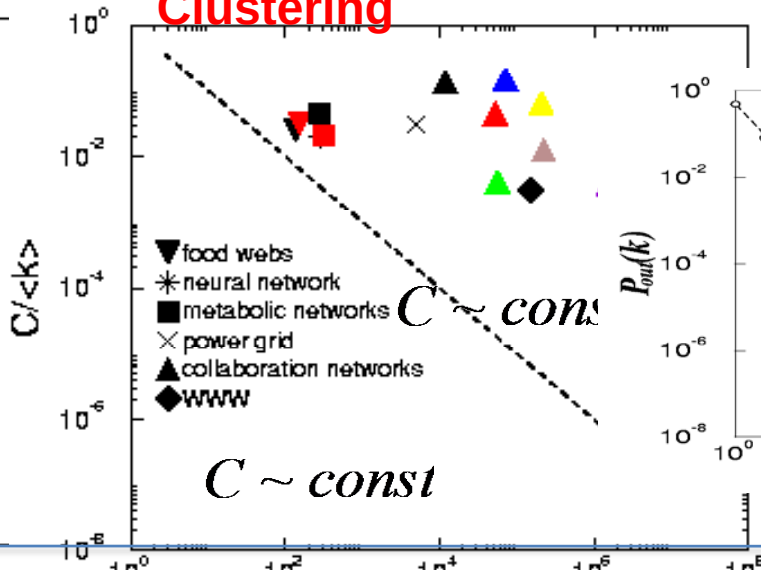
YEP

EMPIRICAL DATA FOR REAL NETWORKS

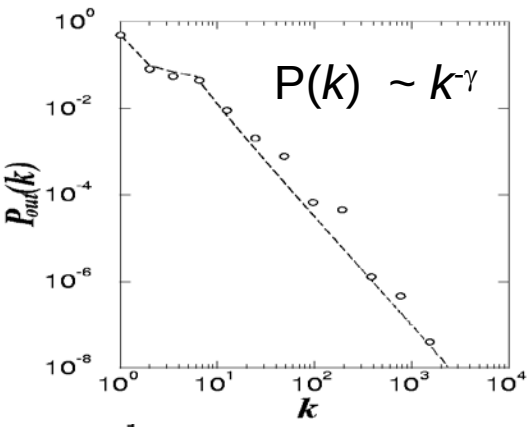
Pathlenght



Clustering

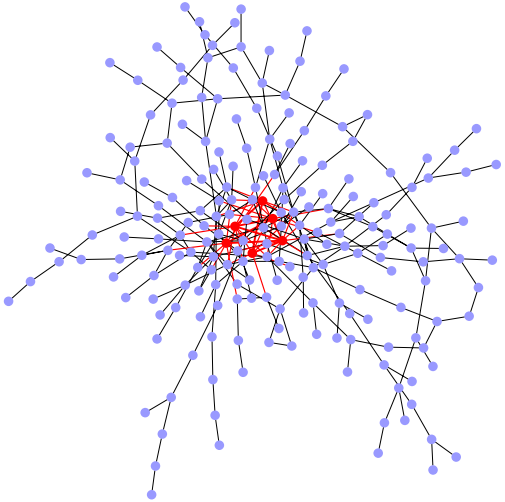


Degree Distr.



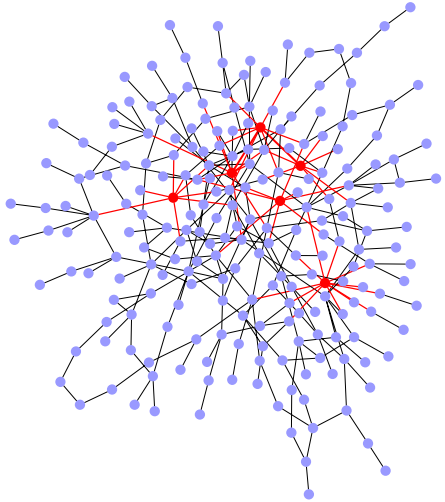
Regular network	$l \approx N^{1/D}$		P(k)=δ(k-k_d)
Erdos-Renyi	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C_{rand} = p = \frac{\langle k\rangle}{N}$	$P(k) = e^{-<k>}\frac{<k>^k}{k!}$
Watts-Strogatz	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C \sim const$	*Exponential*
Barabasi-Albert			$P(k) \sim k^{-\gamma}$

DEGREE CORRELATIONS IN NETWORKS



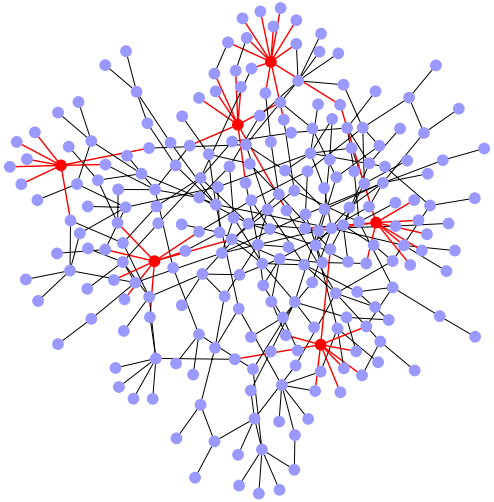
Assortative:

hubs show a tendency to link to each other.



Neutral:

nodes connect to each other with the expected random probabilities.



Disassortative:

Hubs tend to avoid linking to each other.

Quantifying degree correlations (three approaches):

→ full statistical description (Maslov and Sneppen, Science 2001)
→ degree correlation function (Pastor Satorras and Vespignani, PRL 2001)
→ correlation coefficient (Newman, PRL 2002)

STATISTICAL DESCRIPTION

e_{jk}: probability to find a node with degree j and degree k at the two ends of a randomly selected edge

$$\sum_{j,k} e_{jk} = 1 \qquad \sum_{j} e_{jk} = q_k$$

q_k: the probability to have a degree k node at the end of a link.

Where: $$q_k = \frac{k p_k}{\langle k \rangle}$$

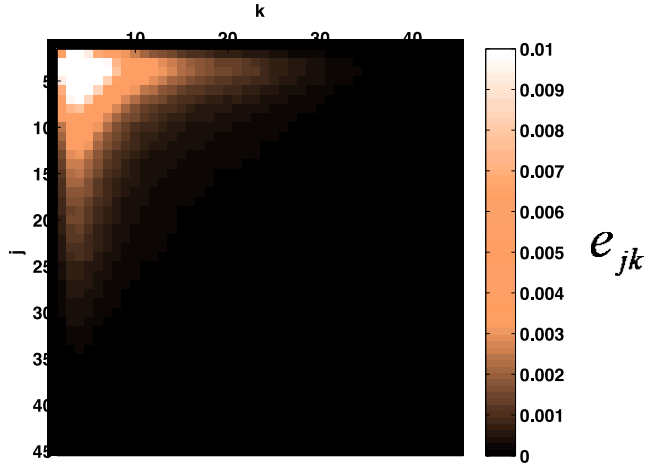
Probability to find a node at the end of a link is biased towards the more connected nodes, i.e. $q_k = C k p_k$, where C is a normalization constant . After normalization we find $C = 1/\langle k \rangle$, or $q_k = k p_k / \langle k \rangle$

If the network has no degree correlations: $$e_{jk} = q_j q_k$$

Deviations from this prediction are a signature of *degree correlation.*

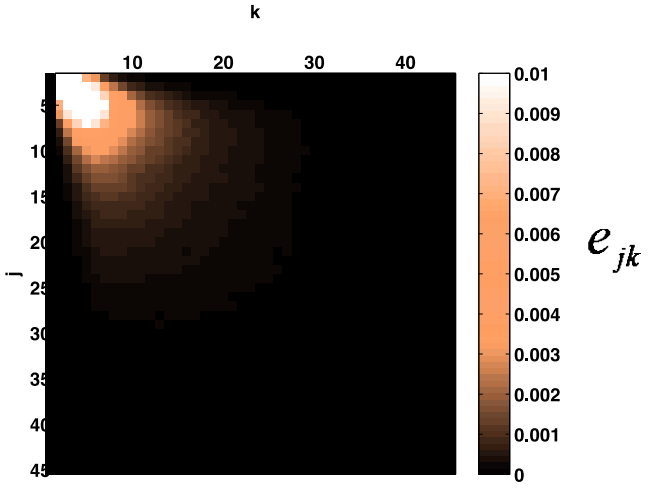
M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)

EXAMPLE: e_{jk} FOR A SCALE-FREE NETWORK

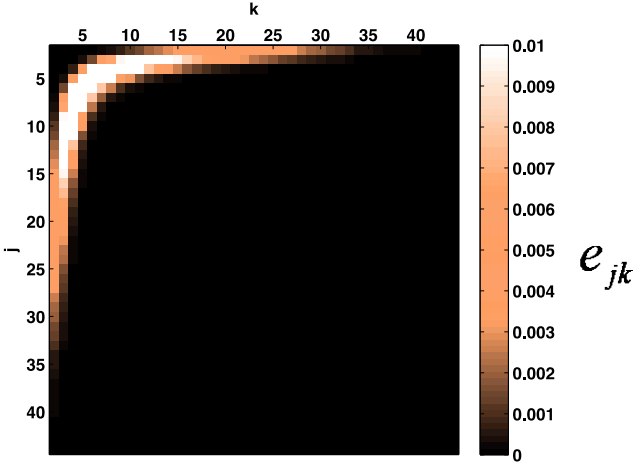


Neutral

Assortative:
More strength in the diagonal, hubs tend to link to each other.



Disassortative:
Hubs tend to connect to small nodes.



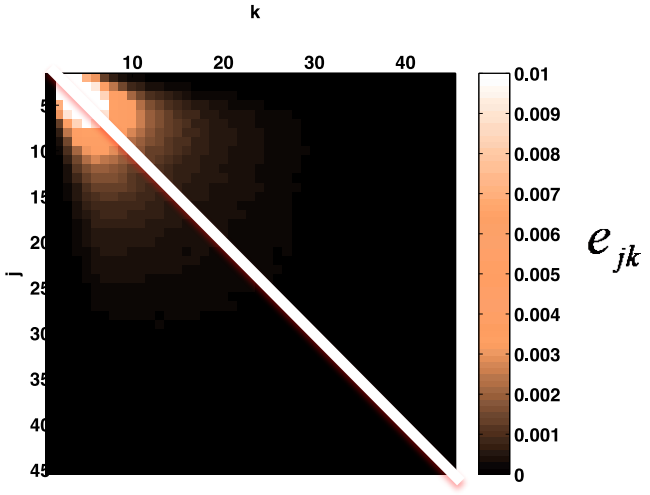
Each matrix is the average of a 100 independent scale-free networks, generated using the static model with $N=10^4$, $\gamma=2.5$ and $\langle k \rangle=3$.

EXAMPLE: e_{jk} FOR A SCALE-FREE NETWORK

Perfectly assortative network:

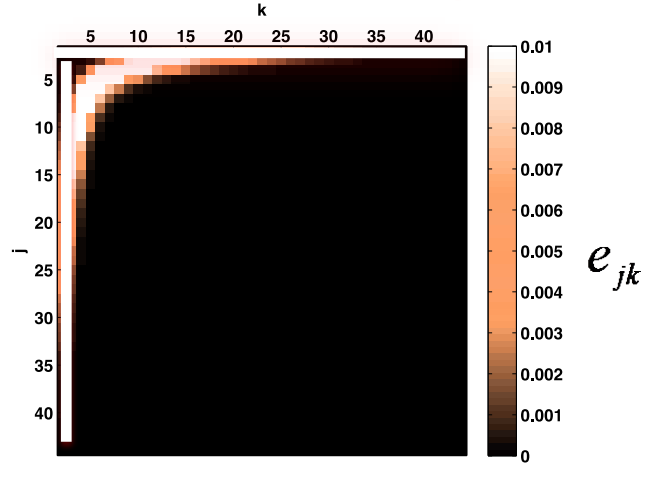
$e_{jk}=q_k\delta_{jk}$

Assortative:
More strength in the diagonal, hubs tend to link to each other.



Perfectly disassortative network:

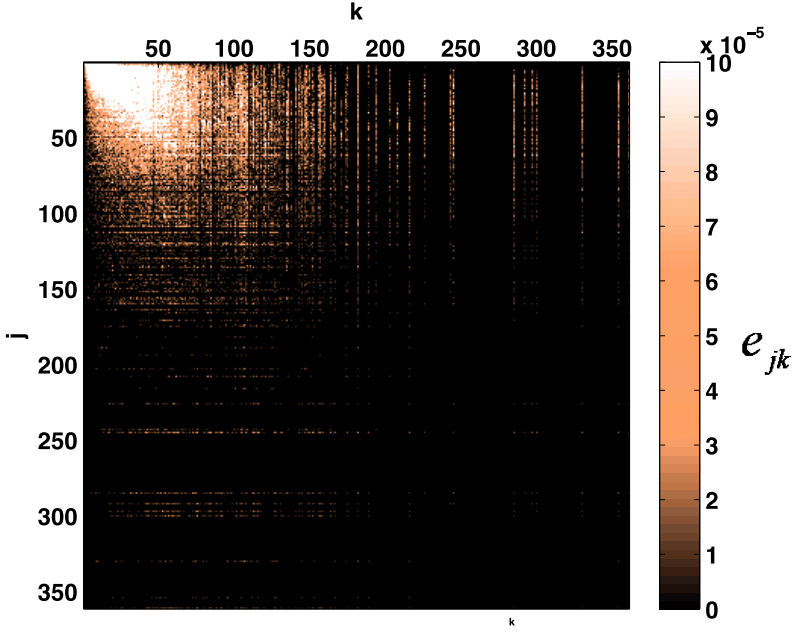
Disassortative:
Hubs tend to connect to small nodes.



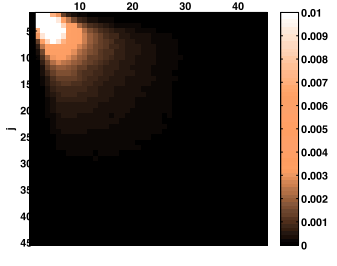
Each matrix is the average of a 100 independent scale-free networks, generated using the static model with $N=10^4$, $\gamma=2.5$ and $\langle k\rangle=3$.

REAL-WORLD EXAMPLES

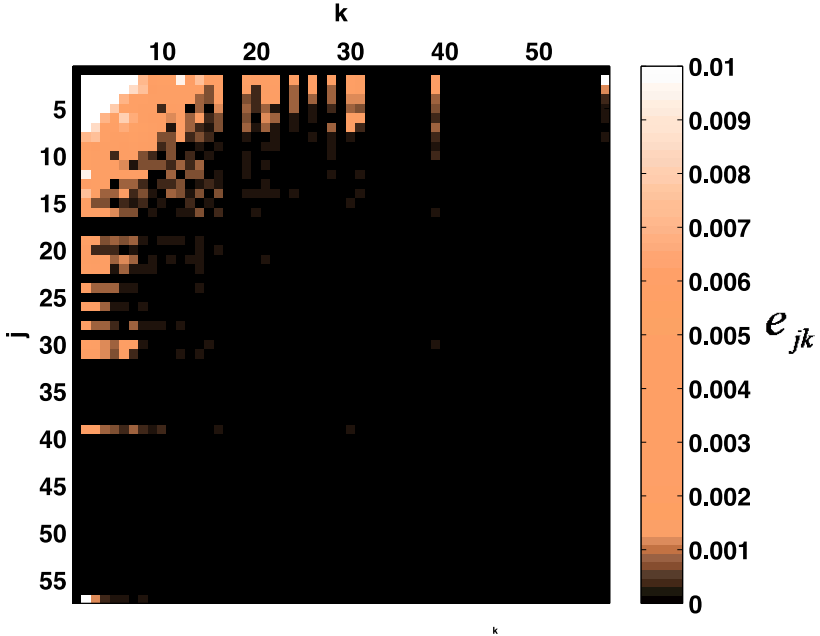
Astrophysics co-authorship network



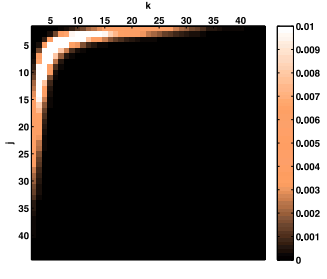
Assortative:
More strength in the diagonal, hubs tend to link to each other.



Yeast PPI

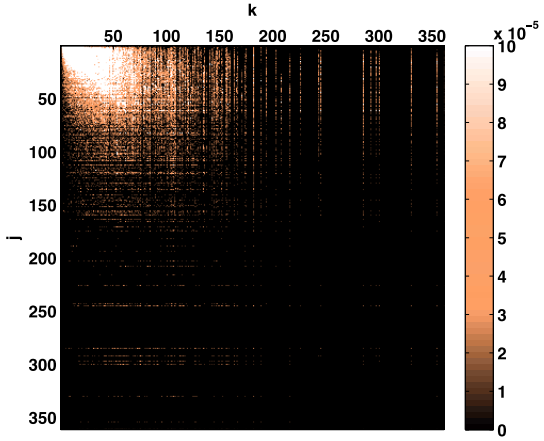


Disassortative:
Hubs tend to connect to small nodes.



PROBLEM WITH THE FULL STATISTICAL DESCRIPTION

(1) Difficult to extract information from a visual inspection of a matrix.



(2) Based on e_{jk} and hence requires a large number of elements to inspect:

$$\frac{k_{\max}\left(k_{\max}-1\right)}{2}-1-k_{\max}$$

Nr. of independent elements

Undirected network: k_{max} x k_{max} matrix

$$\sum_{j,k} e_{jk}=1 \qquad \sum_{j=1,k_{\max}} e_{jk}=q_k$$

Constraints

We need to find a way to reduce the information contained in e_{jk}

M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)