Lecture 7 - Tree Enumeration and Graceful Graphs

Saturday, February 7, 2026 7:08 PM

SLPT: Knee pain causality

- We all have tight hamstrings and hip flexors/quads (rectus femoris) from sitting
- Likewise, most people also have very poor glute activation (even athletes)
- General knee pain (outside of injury) is usually caused by one of the above

Fixes:

- Stretching of hamstrings and hip flexor (get deep into those couch stretches)
- Training proper glute activation (gluteus medius, specifically)
  - Slota's favorite: mini-squats (<1 foot of range, stairs work well)
  - Focus solely on glute activation, don't use your quads!

Story time

w/ Slota

aka "realities of academic writing (proofs)"

Combinatorial

Scientific computing

↳ solve graph probs fast

Has a very defined notion of "quality"

(how fast we solve)

Q: In practice, how much does "being the best" actually matter?

→ Very little, unfortunately

Reason: people evaluating are "human"

→ time constraints, personal issues, fatigue, mistakes

What really matters:

→ Making things "idiot proof"

Note: they aren't idiots

Idiot proof:

- Minimize complexity of presentation
- Stick to common formatting, terminology, basic practices
- Make it look "good", in terms of sentence structure, formatting, general presentation

general presentation

↳ also much easier to read

Takeaway: as many famous writers/speakers have said, "I lacked the time to write a shorter letter"

Now: Graph Theory

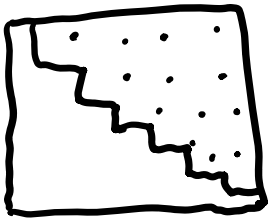
TA/Mentors are grading 100s of proofs

→ think of practical consequences

Your goal: recognize the above and do what you actually need to do, not what you wish you would need to do in a perfect reality. □

Enumerating Trees

Q: How many possible graphs are there on n vertices? (simple, undirected)

Adj matrix  → $\frac{n(n-1)}{2}$ possible edges

↓

either {0, 1}

A: → $2^{\frac{n(n-1)}{2}}$

Q2: How many possible trees?

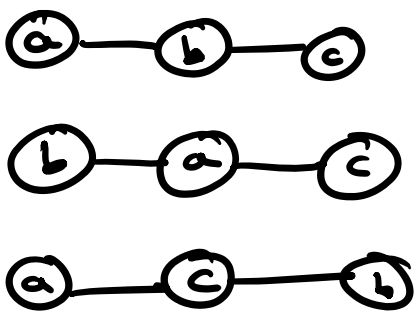
A2: $n^{(n-2)}$ → Cayley's Formula

$n = 2$ → a — b $\quad 2^{2-2} = 1$

$V = \{a, b\}$

$n = 3$ →

$V = \{a, b, c\}$



$3^{2-1} = 3$

→ We'll prove this via Prüfer codes

Prüfer code: a sequence of vertex labels for some T s.t. the length of the sequence is $(n-2)$ and is comprised of T's n distinct vertex labels

$A = \{a_1 a_2 \cdots a_{n-2}\}$ ← Prüfer code

$S = \{\text{vertex labels of } T\}$

$a_i \in S$ ↑ sortable

Q3: How do we construct a Prüfer code from some T?

CreatePrüfer(T):

$A = \phi$ ← empty

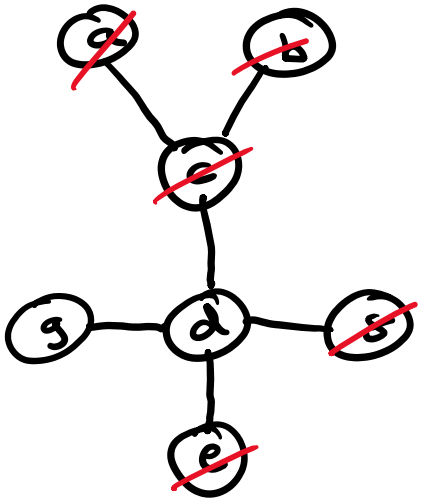
for $i = 1 \ldots (n-2)$

l = label of the "least" remaing leaf

$T = T - l$

a_i = remaing neighbor of l

$A \leftarrow a_i$ append



$S = \{a\ b\ c\ d\ e\ f\ g\}$

$A = \{c\ c\ d\ d\ d\}$

Q4: Given Prüfer code A and vertex labels S, how can we construct a tree?

CreateTree(A, S):

$V(T) = S$

$E(T) = \phi$

$V(T) = S$

$E(T) = \phi$

consider all $s_i \in S$ as "unmarked"

for $i = 1 \ldots (n-2)$

x = least unmarked $s_i \in S$ that is not in $a_i \ldots a_{n-2}$

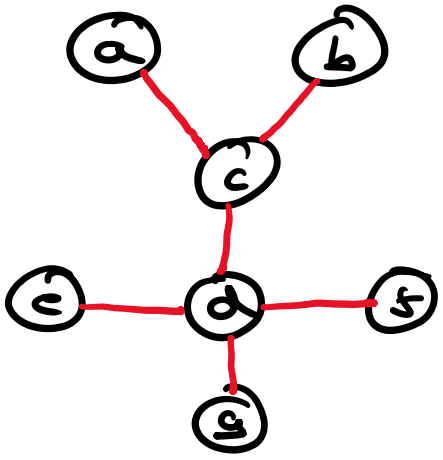
mark x in S

$E(T) \leftarrow (x, a_i)$

(x, y) = remaining unmarked in S

$A = \{c\ c\ d\ d\ d\}$

$S = \{a\ b\ c\ d\ e\ f\ g\}$



Note: for a given tree T and vertex set S, we'll have a unique code A

Given code A and set S, we can construct a unique T

$f(T) = A$ or $f'(A) = T$

$f(T) = A$ or $f'(A) = T$

bijection

Q5: How's this relate back to Cayley's Formula?

A5: There are $n^{(n-2)}$ ways to write a unique Prüfer code

$A = \{a_1 a_2 \cdots a_{n-2}\}$

$a_i \in 1 \ldots n$ possible values

Proofer code Prüf

aka prooving Cayley

What to show: existence and uniqueness of the bijection between all possible trees and Prüfer codes

We'll do strong induction on $n = |S|$

Basis $P(2) \Rightarrow S = \{a\ b\}$

T ⓐ—ⓑ

$f(T) = A = \{\}$

$E(T) = \{(a, b)\}$

1 (a—b)

$E(T) = \{(a, b)\}$

Consider $P(n) = T$

→ Tree T with $V(T) = S$, $|S| = n$

Consider some x as the least value in S where x is a leaf in T

Consider a as the neighbor of x

Construct $P(k) = T' = T - x$

$S' = S - x$

↳ via I.H.: $A' = \{a_2 \dots a_{n-2}\}$

↳ A' exists and is unique for a given (S', T')

Undo castruction $P(k) \rightarrow P(n)$

$T' \rightarrow T$ we add back vertex x

$S' \rightarrow S$ we add back label x

$A' \rightarrow A = \{a \{A'\}\}$

=> From our Prüfer code algorithm,

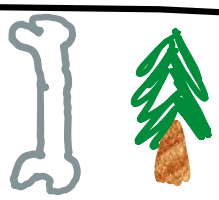
the vertex x and edge (x, a) would be first selected for removal, so the first value in A is guaranteed to be $\underline{a}$

a: neighbor of x

$f(T) = A$ exists for T, S and is unique $\square$

↳ so Cayley's Formula holds ✓

Spanning Trees



Spanner tree

Spanning subgraphs: $S \subseteq G$ is a subgraph s.t. $V(S) = V(G)$

Spanning trees: a spanning subgraph that is a tree

$\tau(G)$ = # of possible spanning trees

$\tau(G)$ = # of possible spanning trees on some graph G

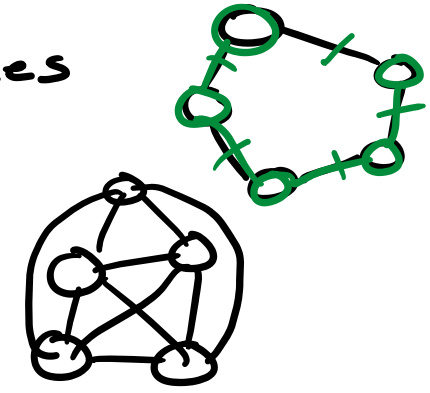
$\tau(G) = 0$ for a disconnected G

$\tau(T) = 1$ for tree T

$\tau(P_n) = 1$ for paths

$\tau(C_n) = n$ for cycles

$\tau(K_n) = n^{(n-2)}$



Q6: How can we count $\tau(G)$ for some arbitrary G

not A6: constructing all trees (hard)

easier A6 for humans: Construct a recurrence relationship (not for computer)

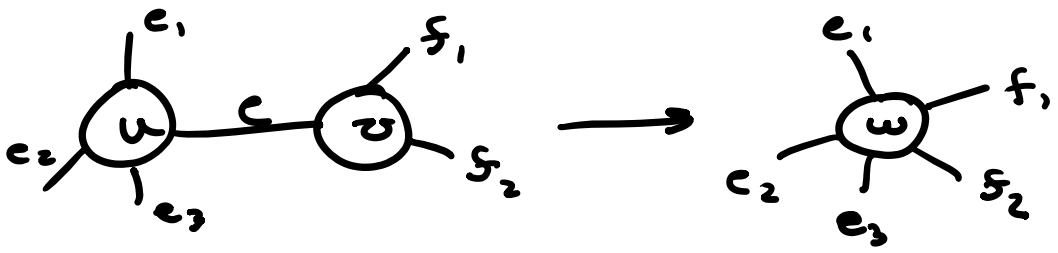
edge contraction

$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

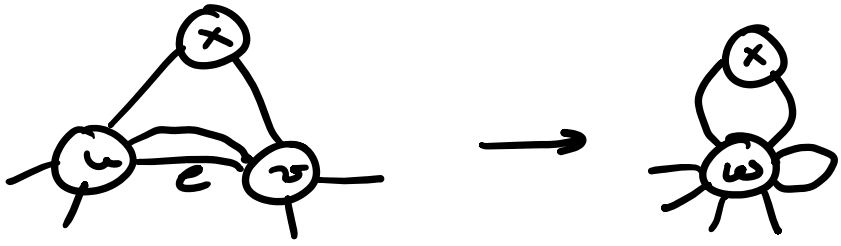
STs — # STs w/o e, $e \in E(G)$ — # STs w/ e

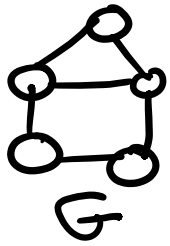
Recall: Edge contractions

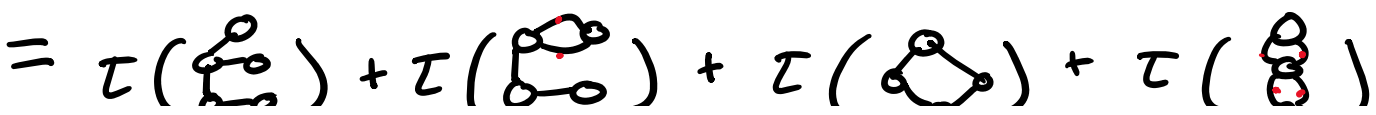
For edge $e = (u, v)$, we combine u, v into a new vertex w that has all of u, v's adjacencies except for edge e

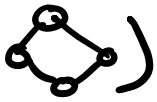


Note: can potentially result in non-simple graph



Example:  → $\tau(G)$?



$= \tau($  $) + \tau($  $) + \tau($  $) + \tau($  $)$

1 2 4 4

$= 1 + 2 + 4 + 4 = 11$