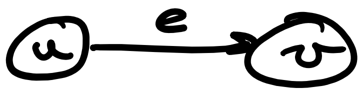
Lecture 6 - Trees

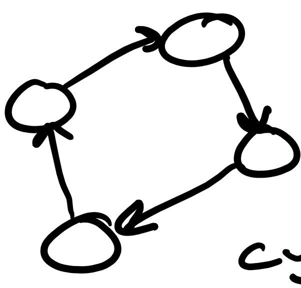
Thursday, February 5, 2026 9:22 AM

Directed graphs

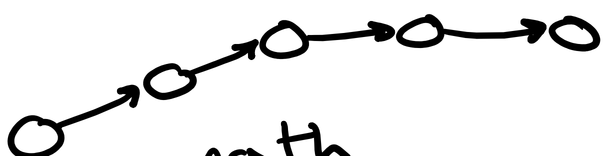


$e = (u, v)$

For walks, trails, paths, cycles
→ same definition as with undirected graphs, but they follow direction of the edges



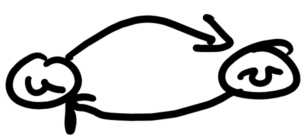
cycle



path

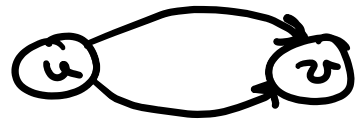
We also have a similar notion for simple, loopy, and multi digraphs

Note: a digraph with edges (u,v) and (v,u) can still be simple



simple

loop



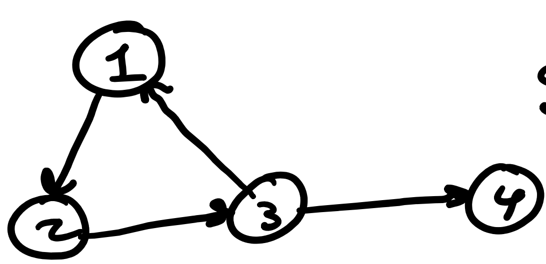
multi-edges

Degree sum formula

$$\sum_{v \in V} d^+(v) = |E| = \sum_{v \in V} d^-(v)$$

Degree sequences

We now consider out and in degree pairs (out, in)



$S = \{(1,1), (1,1), (2,1), (0,1)\}$

Q: How can we tell if a degree sequence is graphic?

Necessary condition:

→ degree sum formula must hold

Q2: Is this condition sufficient?

For simple digraphs → No

For loopy multi-digraphs → Yes

PROOF BY ALGO.

degree pairs in S

consider $(d_i^+, d_i^-) : 1 \le i \le n$

$m = \sum d_i^+ = \sum d_i^- : 1 \le i \le n$

consider m lines

d_i^+ dots get labeled i

d_i^- dots get labeled $-i$

consider n vertices $1 \ldots n$

$v_1 \ldots v_n$

For each line

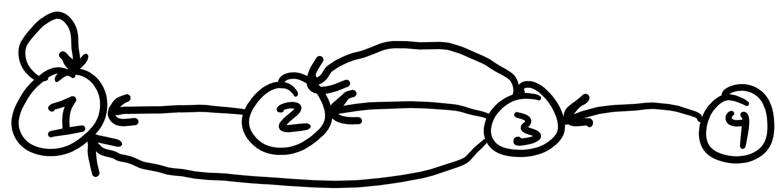
construct an edge as a positive label to negative label corresponding to the vertex set

$$\begin{array}{ccccc} & i=1 & i=2 & i=3 & i=4 \\ S = \{ & (1,3), & (1,2), & (3,1), & (1,0)\} \end{array}$$

$$\begin{array}{lcccccccl} & 1 & 2 & 3 & 3 & 3 & 4 & \text{outs} \\ m \text{ "lines"} & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & 1 & 1 & 1 & 2 & 2 & 3 & \text{in} \end{array}$$

$E = \{(1,1),(2,1),(3,1),(3,2),(3,2),(4,3)\}$

vertex labels



Q: Can we produce all possible realizations of a given degree sequence?

→ equivalent to asking if we can construct all permutations of "out" and "in" labels for our edge list construction process

⇒ yes □

(trivially)

Eulerian Digraphs

↳ has a closed directed trail containing all edges

<u>Necessary</u>: at most 1 nontrivial component

$$\forall v \in V : d^+(v) = d^-(v)$$

Q: are these conditions sufficient?

A: Yes

Proof: same logic as our undirected proof

→ specify a cycle must exist

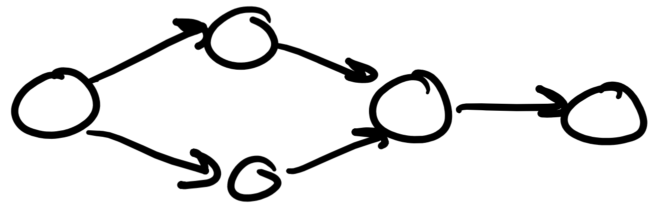
→ inductive proof to construct a directed Euler Tour

Trees

Trees: a connected acyclic undirected simple graphs

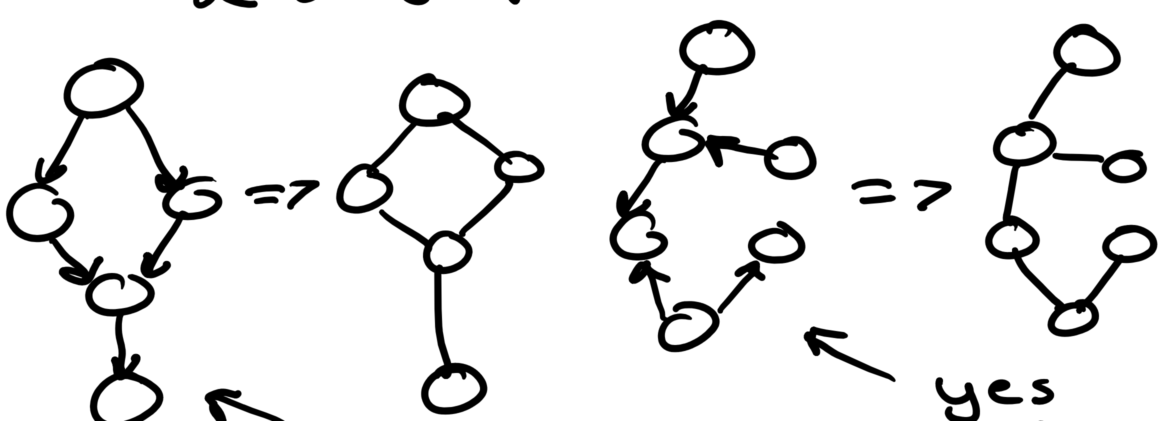
Forest: undirected acyclic graphs

DAG: directed acyclic graphs



Polytree: a DAG where the <u>underlying graph</u> is a tree

underlying graph: undirected graph when we remove edge directions in a digraph



<u>not</u> polytree

<u>yes</u> polytree

Tree T necessary conditions

T is <u>minimally</u> connected

$\forall e \in E(T)$: $T - e$ is disconnected

T is <u>maximally</u> acyclic

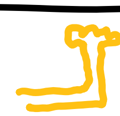
$\forall e \notin E(T)$: $T + e$ is cyclic

T has $|E(T)| = |V(T)| - 1$

T has a single u,v-path
between all $u,v \in V(T)$

T is bipartite

Braincercize: which of the above are also sufficient?

Weak induction  ← arm

(on trees)

Prove: T is a tree $\Rightarrow$ T is bipartite

weak induction on $|V(T)|$ or $|E(T)|$

weak induction on $|V(T)|$ & $|E(T)|$

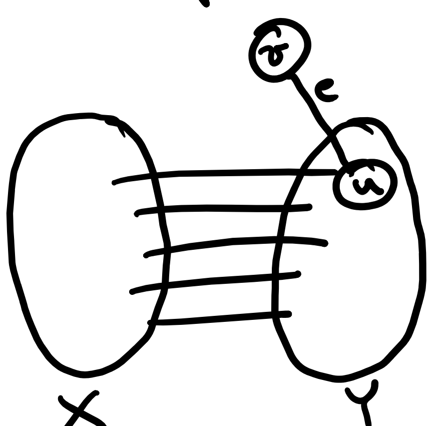
Basis: $P(0) \to 0$ ← single vertex, trivially bipartite

Assume we have tree $P(k)$

↳ by I.H. $P(k)$ has a bipartition

$P(k+1) = P(k) + e$, where e attaches a new leaf to an existing vertex in $P(k)$

Note: we can assume we have a bipartitioning on $P(k)$



$P(k)$

$e = (u, v)$

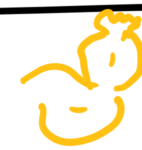
where $u \in V(P(k))$

$v \notin V(P(k))$

w.l.o.g. $u \in Y$

→ we can place the new leaf v in X

=> As v is attached only to u and $v \in X, u \in Y$, and the and $v \in X, u \in Y$, and the rest of $P(k)$ is unaffected, we have a bipartitioning on $P(k+1)$ $\square$

Strong induction  ← arm

Prove: T is a tree $\rightarrow \forall u,v \in V(T)$:

$\exists$ a unique u,v-path

Strong induction on $|V(T)|$

Basis $P(1) = 0$ ✓

Consider tree $P(n)$

Construct $P(k) = P(n) - \ell$, where ℓ is a leaf vertex

Our construction to produce $P(k)$ will not introduce a cycle and will not disconnect the tree

→ I.H. on $P(k)$ to give us unique

↳ I.H. on $P(k)$ to give us unique u,v-paths $\forall u,v \in V(P(k))$

Bring it on back to $P(n)$

→ we add back our leaf ℓ

Bring it on home

→ by showing our result via I.H. still holds on $P(n)$

We already know:

$$\left\{ \begin{matrix} \forall u,v \in V(P(n)) \\ u,v \neq \ell \end{matrix} : \exists \text{ unique } u,v\text{-path} \right\}$$

We want to define our u,ℓ-paths

→ consider x where $x \in N(\ell)$

→ we have unique u,x-paths

→ we construct u,ℓ-paths by taking all unique u,x-paths and adding the edge (x,ℓ)

=> this constructs all of our u,ℓ-paths $\square$

$\Rightarrow$ This constructs all of our

u, ℓ-paths $\square$

Note: weak induction logic is often equivalent to strong induction on trees

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Extra fun definitions

(for general graphs)

distance: $d(u,v)$ = length of shortest u,v-path

diameter: $D(G)$ = length of the longest u,v-path

lil' epsilon

eccentricity: $\varepsilon(v) = \max d(u,v)$ $\forall u \in V(G)$

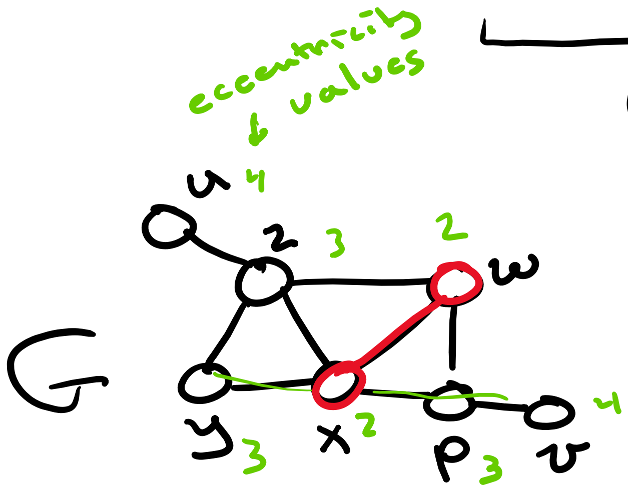
radius: $R(G) = \min \varepsilon(u)$ $\forall u \in V(G)$

Center of G = induced subgraph of vertices with the minimum eccentricity of G

$\{\forall u \in V(G) : \varepsilon(u) = R(G)\}$

$$\underbrace{\{\forall u \in V(G) : \varepsilon(u) = R(G)\}}$$

↳ center is the subgraph induced on this set of vertices



$d(u,v) = 4$

$D(G) = 4$

$\varepsilon(w) = 2$

$R(G) = 2$

$\{w, x\}$ is center

↳ induce subgraph

Note: center is not necessarily connected