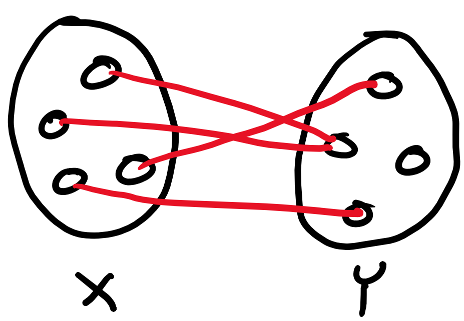
Lecture 2 - Graph Classes and Isomorphism

Thursday, January 15, 2026 8:47 AM

"Graph Theory is a terminological jungle, in which any newcomer may plant a tree."
- John Barnes

Bipartite graphs:

Graphs whose vertices can be separated into two vertex-disjoint set, s.t. between any u, v
(such that)
within one set, there are no edges



← no edges within a set

Biclique : a.k.a. complete bipartite graphs
(also known as)

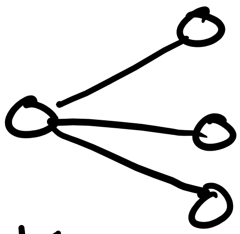
a bipartite graph with X, Y sets with all $x \in X$ connecting to all $y \in Y$

$\forall x \in X, \forall y \in Y: (x, y) \in E$

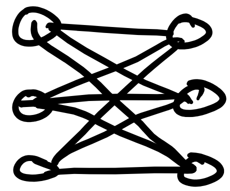
$\rightarrow \forall x \in X, \forall y \in Y: (x, y) \in E$

Notation: $K_{i,j}$

$\uparrow |X| \quad \uparrow |Y|$



$K_{1,3}$



$K_{3,3}$

Graph Representations

Adjacency matrix

(don't worry about CSR, adjacency list or incident matrix)

$\rightarrow$ Matrix A: an $n \times n$ matrix which represents the structure of some G, $|V(G)| = n$

If edge $(i, j) \in E(G)$

$\rightarrow a_{i,j}$ is nonzero

If edge $(i, j) \notin E(G)$

$\uparrow$ not in

if edge $(i,j) \notin E(G)$

not in

$$\rightarrow a_{i,j} = 0$$

assume $V(G) = \{1, 2, \ldots, n\}$

For undirected simple graphs:

$a_{i,j} \in \{0, 1\}$ edge exists or not

$a_{ii} = 0$ no loops

$a_{j,i} = a_{i,j}$ edges are symmetric

For multigraphs:

$a_{i,j} \geq 0 \rightarrow$ value is dependent on # of (i,j) edges

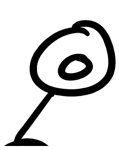
For loopy graphs:

$a_{ii} \in \{0, 2\} \rightarrow$ value of 2 indicates a self loop

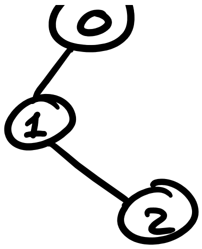
For loopy multigraphs:

$a_{i,j} \geq 0$

$a_{i,i} = \{0, 2, 4, \ldots\} \rightarrow$ value is 2x the # of self loops

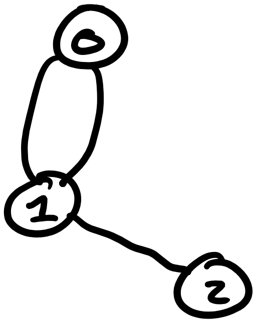


$$\begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ 0 & \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \end{array}$$



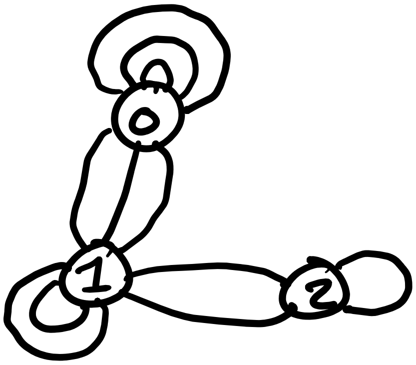
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & & & \\ 1 & & & \\ 2 & & & \end{array}\quad \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\sum row_i = d(i)$$



$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

$$\sum row_1 = 3 = d(1)$$

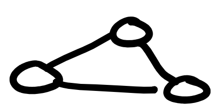


$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array}\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 6 & 3 & 0 \\ 3 & 4 & 2 \\ 0 & 2 & 2 \end{bmatrix} \end{array}$$

$$\sum row_1 = 9 = d(1)$$

Graph Properties and Classes

Property: a description or characteristic that some graph holds

Consider K_3: 

Property: $|V(K_3)| = |E(K_3)| = 3$

Property: K_3 is cyclic

Property: K_3 is cyclic
(has a cycle)

Property: $\forall v \in V(K_3) : d(v) = 2$

aka K_3 is <u>2-regular</u>

<u>k-regular</u> graph G has
$\forall v \in V(G) : d(v) = k$

<u>Graph class</u>: a set of all possible graphs having the same set of properties

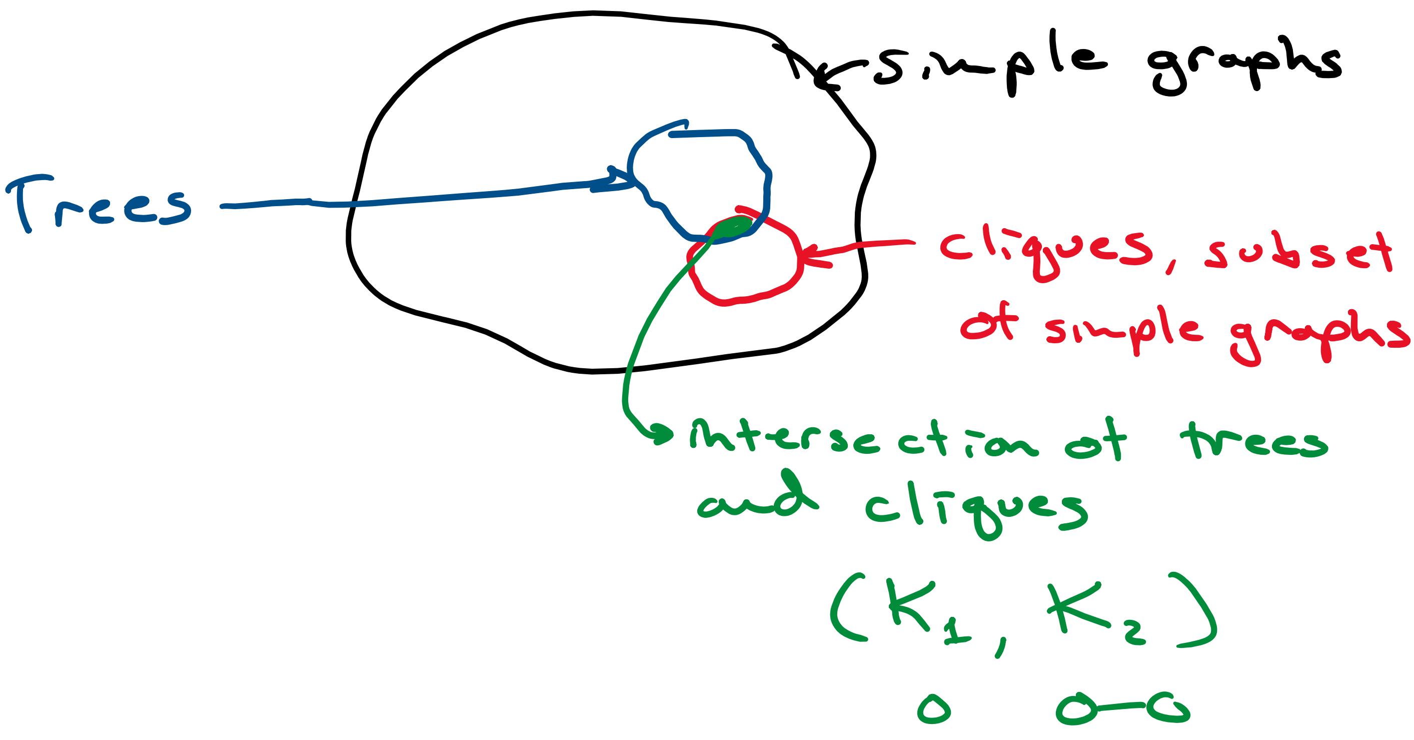
Examples:

simple graphs $\rightarrow$ P_1: no multiedges
P_2: no self loops

clique graphs $\rightarrow$ P_1: simple
P_2: all possible edges

Note: graph classes all can be defined based on some subset / superset / intersection relationship

relationship



Graph Isomorphism

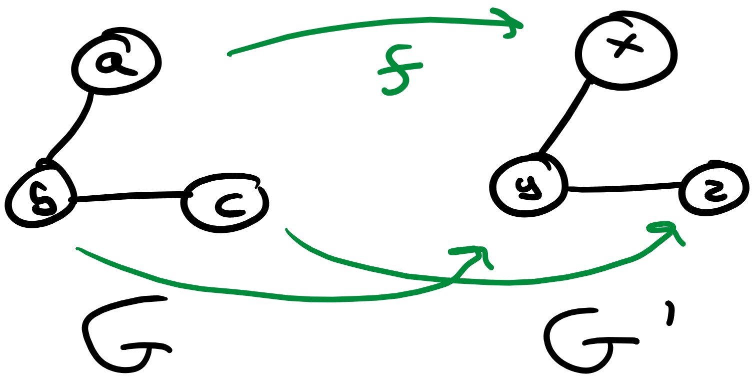
Graph $G = \{V, E\}$ is isomorphic to $G' = \{V', E'\}$ if there exists a bijective function f

$$f : V \to V'$$

$$\forall v \in V \to f(v) = u \in V'$$

$$\forall e = (a, b) \in E \to (f(a), f(b)) \in E'$$

$\forall e = (a,b) \in E \rightarrow (f(a), f(b)) \in E'$



$f(a) = x$

$f(b) = y$

$f(c) = z$

$(a,b) \rightarrow (f(a), f(b)) = (x,y) \in E'$ ✓

$(b,c) \rightarrow (f(b), f(c)) = (y,z) \in E'$ ✓

Notation: $G \cong G'$

isomorphic

All isomorphic graphs belong to the same <u>isomorphism class</u>

If $G \cong G'$:

the sorted <u>degree sequence</u> is going to be equivalent

list of all degrees

the <u>diameters</u> will be equal

The diameters will be equal

↑ length of the longest shortest path

the girths will be equal

↑ shortest cycle length

all constituent subgraphs will be identical

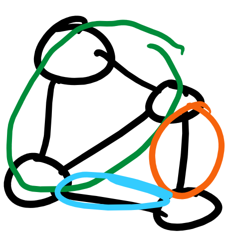
→ subgraph $H \subseteq G$

$V(H) \subseteq V(G)$

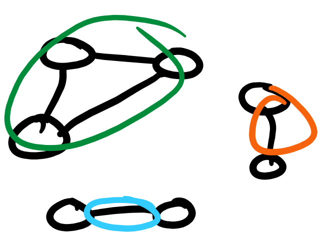
$E(H) \subseteq E(G)$

And, a decomposition on G also equivalently exists on G'

→ deconstructing G into edge-disjoint subgraphs in a bijective fashion



G



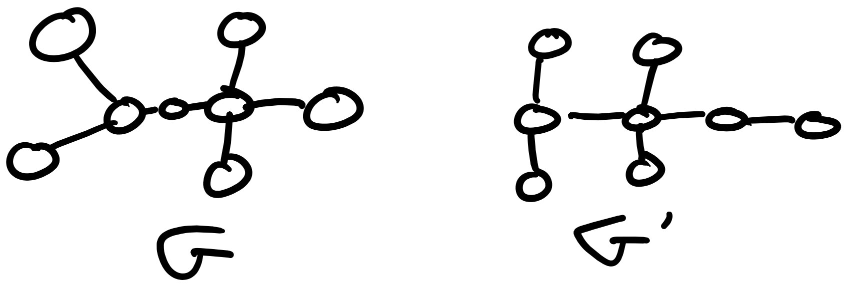
decomposition of G

Note: the properties

above are all

<u>Necessary</u> but not

<u>sufficent</u> to prove isomorphism



$S = \{1, 1, 1, 1, 1, 2, 4, 3\}$ $S' = \{1, 1, 1, 1, 1, 2, 4, 3\}$

To disprove G and G' are not isomophic → simply show two topological properties are not equivalent

To prove $G \cong G'$ are isomorphic: you need to explicitly identify and validate that bijective mapping f

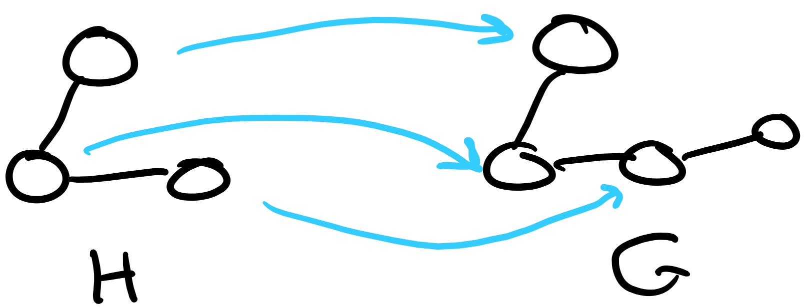
mapping f

→ much harder to do

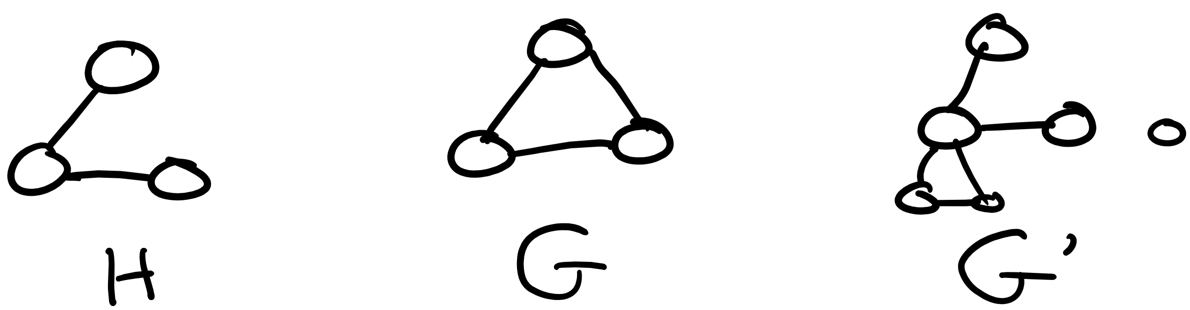
Subgraph Isomorphism

Is determining whether some H is a subgraph of some G

$$V(H) \subseteq V(G)$$
$$E(H) \subseteq E(G)$$



Induced vs. non-induced



H is an <u>induced</u> subgraph

H is an induced subgraph
of G' only
H is a non-induced subgraph
of both G and G'

Induced subgraph H of G:

consider our vertex set of H mapped
to some vertices in G
- all edges between vertices in H will
have edges between mapped
vertices in G
- all missing edges between vertices
in H will not have edges
between mapped vertices in G

don't care

Non-induced: we don't care about
missing edges in H vs. G