Foundations of Computer Science
Lecture 23

Languages: What is Computing?

A Formal Model of a Computing Problem
Decision Problems and Languages
Describing a Language: Regular Expressions
Complexity of a Computing Problem

"Say what's on your mind, Harris—the language of dance has always eluded me."

Last Time

1. Comparing infinite sets.
2. Countable.
   - $\mathbb{N}_0, E, \mathbb{Z}, \mathbb{Q}$ are countable.
   - Finite binary strings $\mathcal{B}$ is countable.
3. Uncountable
   - *Infinite* binary strings are uncountable.
   - Reals are uncountable.
4. Infinity and computing.
   - Programs are finite binary strings (countable).
   - Functions we might like to compute are infinite binary strings (uncountable).
   - Conclusion: there are **MANY** functions which *cannot* be computed by programs.

Today: Languages: What is Computing?

1. Decision problems.

2. Languages.
   - Describing a language.

3. Complexity of a computing problem.

What is a Computing Problem?

Decide YES or NO whether a given integer $n \in \mathbb{N}$ is prime.

What is a Computing Problem?

Decide YES or NO whether a given integer $n \in \mathbb{N}$ is prime.

List the primes in increasing order (primes are countable),

$$\text{primes} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$$

What is a Computing Problem?

Decide YES or NO whether a given integer $n \in \mathbb{N}$ is prime.

List the primes in increasing order (primes are countable),

$$\text{primes} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$$

Given $n \in \mathbb{N}$, walk through this list.

1: If you come to n output YES.
2: If you come to a number bigger than n, output NO.

Not the smartest approach to primality testing, but gets to the heart of computing

What is a Computing Problem?

Decide YES or NO whether a given integer $n \in \mathbb{N}$ is prime.

List the primes in increasing order (primes are countable),

$$\text{primes} = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots\}$$

Given $n \in \mathbb{N}$, walk through this list.

1: If you come to n output YES.
2: If you come to a number bigger than n, output NO.

Not the smartest approach to primality testing, but gets to the heart of computing

LANGUAGES

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}.$$

The light is on for 1010 pushes, if and only if $1010 \in \mathcal{L}_{\text{push}}$.

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}.$$

The light is on for 1010 pushes, if and only if $1010 \in \mathcal{L}_{\text{push}}$.

The door should open if a person is on the mat.
Walk on (1) or off (0). E.g. 10110 means on, off, on, on, off $\rightarrow$ open.

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}.$$

The light is on for 1010 pushes, if and only if $1010 \in \mathcal{L}_{\text{push}}$.

The door should open if a person is on the mat.
Walk on (1) or off (0). E.g. 10110 means on, off, on, on, off $\rightarrow$ open.

$$\mathcal{L}_{\text{door}} = \{1, 11, 101, 110, 111, 1011, 1101, 1110, 1111, \ldots\}.$$

Given input w, e.g. $w = 1011$, the door is open if and only if $w \in \mathcal{L}_{\text{door}}$.

Decision Problems

$$\mathcal{L}_{\text{prime}} = \{10, 11, 101, 111, 1011, 1101, 10001, 10011, 10111, 11101, \ldots\}. \qquad \text{(primes in binary)}$$

9 is prime $\leftrightarrow$ the *string* 1001 is in $\mathcal{L}_{\text{prime}}$.

The light is off. Every push toggles between on and off.
Given the number of pushes, decide whether the light is on or off.
Encode number of pushes by a binary string, e.g. 101 means 5 pushes.

$$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}.$$

The light is on for 1010 pushes, if and only if $1010 \in \mathcal{L}_{\text{push}}$.

The door should open if a person is on the mat.
Walk on (1) or off (0). E.g. 10110 means on, off, on, on, off $\rightarrow$ open.

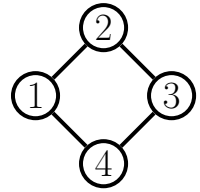
$$\mathcal{L}_{\text{door}} = \{1, 11, 101, 110, 111, 1011, 1101, 1110, 1111, \ldots\}.$$

Given input w, e.g. $w = 1011$, the door is open if and only if $w \in \mathcal{L}_{\text{door}}$.

Decision problems can be formulated as testing membership in a set of strings

A Decision Problem on Graphs

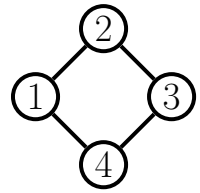
(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**



A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

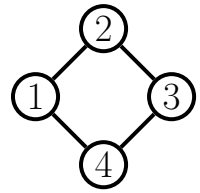
(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.

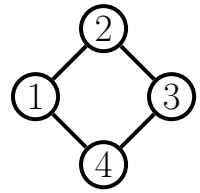


(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



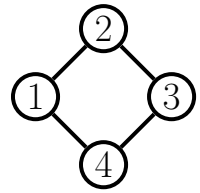
(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

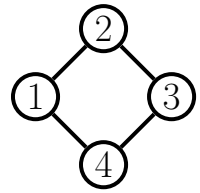
Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

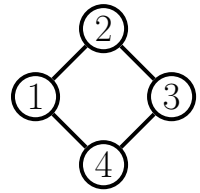
" $1, 2, 3, 4$ "

nodes

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

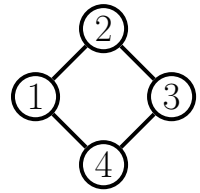
" $1, 2, 3, 4 \mid (1, 2)(2, 3)(3, 4)(4, 1)$ "

nodes edges

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

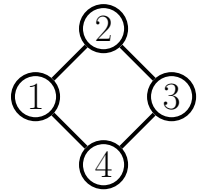
" $1, 2, 3, 4 \mid (1, 2)(2, 3)(3, 4)(4, 1) \mid 1, 3$ "

nodes — edges — endpoints of path

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

$$\text{“ } 1,2,3,4 \mid (1,2)(2,3)(3,4)(4,1) \mid 1,3 \mid 3 \text{ ”}$$

nodes — edges — endpoints of path — target distance

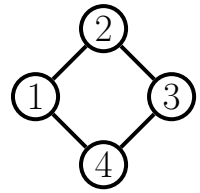
The graph problem can be encoded as a binary string using ASCII

00110001001011000011001000101100001100110010110000110100011111000010100000110001001011000011001000101001001010000011001000101100001100110010100100101000001100100010110000110011001010010010100000110011001011000011010000101001001010000011010000101100001100010010100101111100001100010010110000110011011111000011001100101001001010000011001100101100001101000010100100101000001101000010110000110001001010010111110000110001001011000011001101111100001100110010100100101000001100110010110000110100001010010010100000110100001011000011000100101001011111000011000100101100001100110111110000110011.

A Decision Problem on Graphs

(a)**[Optimization]** What's distance between nodes ① and ③? Answer: **2**

(b)**[Decision]** Is there a path between ① and ③ of length at most 3? YES.



(a) is harder than (b): (a)'s answer gives (b)'s answer instantly.

Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance.

"Is there a path of length at most 3 between nodes ① and ③ in the graph above."

becomes

$$\text{“ } 1,2,3,4 \mid (1,2)(2,3)(3,4)(4,1) \mid 1,3 \mid 3 \text{ ”}$$

nodes — edges — endpoints of path — target distance

The graph problem can be encoded as a binary string using ASCII

00110001001011000011001000101100001100110010110000110100011111000010100000110001001011000011001000101001001010000011001000101100001100110010100100101000001100110010110000110100001010010010100000110100001011000011000100101001011111000011000100101100001100110111110000110011.

$$\mathcal{L}_{\text{path}} = \left\{ \begin{array}{l} \text{All strings of the form “nodes | edges | endpoints of path | target distance” for which} \\ \text{the distance between the endpoints in the graph is at most the target distance.} \end{array} \right\}$$

Pop Quiz. YES or NO: “ $1,2,3,4,5 \mid (1,2)(2,3)(3,5)(3,4) \mid 1,5 \mid 2$ ”

Is Optimization Really Harder than Decision?

Is Optimization Really Harder than Decision?

If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes (x) and (y) of length at most 1? NO

Is Optimization Really Harder than Decision?

If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes (x) and (y) of length at most **1**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **2**? NO

Is Optimization Really Harder than Decision?

If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes (x) and (y) of length at most **1**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **2**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **3**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **4**? YES

You ask the decision question until the answer is YES.

The minimum-pathlength between (x) and (y) is 4.

It can take long, but it works.

Is Optimization Really Harder than Decision?

If you can solve the decision problem, you can solve the optimization problem.

Is there a path in the graph between nodes (x) and (y) of length at most **1**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **2**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **3**? NO
Is there a path in the graph between nodes (x) and (y) of length at most **4**? YES

You ask the decision question until the answer is YES.

The minimum-pathlength between (x) and (y) is 4.

It can take long, but it works.

Decision and optimization are "equivalent" when it comes to *solvability*.

A computing problem is a decision problem.

Languages

Standard formulation of a decision problem:

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D
Input: Finite graph G; nodes x, y; target distance D.

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D
Input: Finite graph G; nodes x, y; target distance D.
Question: Is there an (x,y)-path in G of length at most D.

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D
Input: Finite graph G; nodes x, y; target distance D.
Question: Is there an (x,y)-path in G of length at most D.

Every decision problem has a YES-set, which we usually don't explicitly list.

$$\begin{aligned} \text{YES-set} &= \{\text{input strings } w \text{ for which the answer is YES}\} \\ &= \{w_1, w_2, w_3, \ldots\}. \end{aligned}$$

← A *language* is any set of finite binary strings

Languages

Standard formulation of a decision problem:

Problem: GRAPH-DISTANCE-D
Input: Finite graph G; nodes x, y; target distance D.
Question: Is there an (x,y)-path in G of length at most D.

Every decision problem has a YES-set, which we usually don't explicitly list.

$$\begin{aligned} \text{YES-set} &= \{\text{input strings } w \text{ for which the answer is YES}\} \\ &= \{w_1, w_2, w_3, \ldots\}. \end{aligned}$$

← A *language* is any set of finite binary strings

A computing problem is a YES-set, a set of *finite* binary strings.

Computing Problems Are Languages

Language: Set of finite binary strings.

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem
Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

$\{\varepsilon, 1, 10, 01\}$ ← finite language

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	$\leftarrow$ all finite strings

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	$\leftarrow$ all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	$\leftarrow$ all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	$\leftarrow$ finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	$\leftarrow$ all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	← finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	← all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	← strings of 1s

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	← finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	← all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	← strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	← finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	← all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	← strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{0^n1^n}$	$\{01, 0011, 000111, \ldots\} = \{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$	

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	← finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	← all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	← strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{0^n1^n}$	$\{01, 0011, 000111, \ldots\} = \{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{\text{pal}}$	$\{\varepsilon, 0, 1, 00, 11, 000, 010, 101, 111, \ldots\}$	← palindromes

Computing Problems Are Languages

Language: Set of finite binary strings.

Solving the problem

Give a "procedure" to tell if a general input w is in the language (YES-set).

Abstract, precise and general formulation of a computing problem.

	$\{\varepsilon, 1, 10, 01\}$	← finite language
Σ^*	$\{\varepsilon, 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, \ldots\}$	← all finite strings
$\mathcal{L}_{\text{prime}}$	$\{10, 11, 101, 111, 1011, 1101, 10001, \ldots\}$	
$\mathcal{L}_{\text{push}}$	$\{1, 01, 11, 001, 011, 101, 111, 0001, 0011, \ldots\}$	
$\mathcal{L}_{\text{door}}$	$\{1, 11, 101, 110, 111, 1011, 1101 \ldots\}$	
$\mathcal{L}_{\text{unary}}$	$\{\varepsilon, 1, 11, 111, 1111, \ldots\} = \{1^{\bullet n} \mid n \geq 0\}$	← strings of 1s
$\mathcal{L}_{(01)^n}$	$\{\varepsilon, 01, 0101, 010101, \ldots\} = \{(01)^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{0^n1^n}$	$\{01, 0011, 000111, \ldots\} = \{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$	
$\mathcal{L}_{\text{pal}}$	$\{\varepsilon, 0, 1, 00, 11, 000, 010, 101, 111, \ldots\}$	← palindromes
$\mathcal{L}_{\text{repeated}}$	$\{\varepsilon, 00, 11, 0000, 0101, 1010, 1111, \ldots\}$	← repeated strings

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Use a variable to formally define $\mathcal{L}$:

$$\mathcal{L} = \{w \mid w = (01)^{\bullet n}, \text{ where } n \geq 0\}. \qquad \text{(informally } \{(01)^{\bullet n} \mid n \geq 0\}\text{)}$$

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Use a variable to formally define $\mathcal{L}$:

$$\mathcal{L} = \{w \mid w = (01)^{\bullet n}, \text{ where } n \geq 0\}. \qquad \text{(informally } \{(01)^{\bullet n} \mid n \geq 0\}\text{)}$$

More than one variable:

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Use a variable to formally define $\mathcal{L}$:

$$\mathcal{L} = \{w \mid w = (01)^{\bullet n}, \text{ where } n \geq 0\}. \qquad \text{(informally } \{(01)^{\bullet n} \mid n \geq 0\}\text{)}$$

More than one variable:

$$\{u \bullet v \mid u \in \Sigma^* \text{ and } v = u^{\mathrm{R}}\} = \{\varepsilon, 00, 11, 0000, 0110, 1001, 1111, \ldots\}. \quad \leftarrow \text{even palindromes}$$

Exercise. Define $\mathcal{L}_{\text{add}} = \{0100, 011000, 001000, 00110000, 00010000, 0001100000, 01110000, 0011100000, 000111000000, \ldots\}$

Describing a Language: String Patterns and Variables

An example where there is a clear pattern,

$$\mathcal{L} = \{\varepsilon, 01, 0101, 010101, \ldots\}.$$

Use a variable to formally define $\mathcal{L}$:

$$\mathcal{L} = \{w \mid w = (01)^{\bullet n}, \text{ where } n \geq 0\}. \qquad \text{(informally } \{(01)^{\bullet n} \mid n \geq 0\}\text{)}$$

More than one variable:

$$\{u \bullet v \mid u \in \Sigma^* \text{ and } v = u^{\mathrm{R}}\} = \{\varepsilon, 00, 11, 0000, 0110, 1001, 1111, \ldots\}. \quad \leftarrow \text{even palindromes}$$

Exercise. Define $\mathcal{L}_{\text{add}} = \{0100, 011000, 001000, 00110000, 00010000, 0001100000, 01110000, 0011100000, 000111000000, \ldots\}$ Ans: $\{0^{\bullet n} \bullet 1^{\bullet m} \bullet 0^{\bullet n+m}\}$

For more complicated patterns, we use regular expressions, e.g. the Unix/Linux command

ls FOCS* (Lists everything that starts with FOCS ($*$ is the "wild-card").)

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement (Familiar.)

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\{0, 01\} \bullet \{0, 11\} \;=\; \{00, 011, 010, 0111\}$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \end{aligned} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0, 01\}^* = \{\varepsilon,$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0, 01\}^* = \{\varepsilon, 0, 01,$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0, 01\}^* = \{\varepsilon, 0, 01, 00, 001, 010, 0101,$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0, 01\}^* = \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0001, 0010, \ldots\} = \bigcup_{n=0}^{\infty} \{0, 01\}^{\bullet n};$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Basic building blocks are finite languages:

$$\{1, 11\} \qquad \{0, 01\} \qquad \{00\} \qquad \{1\}$$

Combine these using

union, intersection, complement	(Familiar.)
concatenation $\bullet$, Kleene-star *	(What?!?)

Concatenation of languages.

$$\mathcal{L}_1 \bullet \mathcal{L}_2 \bullet \mathcal{L}_3 = \{w_1 \bullet w_2 \bullet w_3 \mid w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}.$$

$$\begin{aligned} \{0, 01\} \bullet \{0, 11\} &= \{00, 011, 010, 0111\} \\ \{0, 11\} \bullet \{0, 01\} &= \{00, 001, 110, 1101\} \qquad \mathcal{L}_1 \bullet \mathcal{L}_2 \neq \mathcal{L}_2 \bullet \mathcal{L}_1 \\ \{0, 01\} \bullet \{0, 01\} = \{0, 01\}^{\bullet 2} &= \{00, 001, 010, 0101\} \qquad \text{(self-concatenation)} \end{aligned}$$

Pop Quiz. What is $\{0, 01\} \bullet \{1, 10\}$? What is $\{0, 01\}^{\bullet 3}$? What is $\{0, 01\}^{\bullet 0}$?

Kleene star: All possible concatenations of a finite number of strings from a language.

$$\{0, 01\}^* = \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0001, 0010, \ldots\} = \bigcup_{n=0}^{\infty} \{0, 01\}^{\bullet n};$$

$$\{1\}^* = \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\} = \bigcup_{n=0}^{\infty} \{1\}^{\bullet n}.$$

Pop Quiz. Which of the strings $\{101110, 00111, 00100, 01100\}$ can you generate using $\{0, 01\}^* \bullet \{1, 10\}^*$?

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

$$\{0, 01\}^* = \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0010, \ldots\}$$
$$\{1\}^* = \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\}$$

To generate 1110111:

$$11 \in \{1, 11\}$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

$$\begin{aligned} \{0, 01\}^* &= \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0010, \ldots\} \\ \{1\}^* &= \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\} \end{aligned}$$

To generate 1110111:

$$11 \in \{1, 11\}$$

$$10 \in \overline{\{0, 01\}^*}$$

The Regular Expression: $\{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

$$\begin{aligned} \{0, 01\}^* &= \{\varepsilon, 0, 01, 00, 001, 010, 0101, 000, 0010, \ldots\} \\ \{1\}^* &= \{\varepsilon, 1, 11, 111, 1111, 11111, \ldots\} \end{aligned}$$

To generate 1110111:

$$11 \in \{1, 11\}$$
$$10 \in \overline{\{0, 01\}^*}$$
$$111 \in \{00\} \cup \{1\}^*$$

Hence $1110111 \in \{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$

Pop Quiz Is there another way to generate 1110111?

Pop Quiz Yes or no: $11110010 \in \{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*)$?

Challenges Involving Regular Expressions

1. Is there a simple procedure to test if a given string satisfies a regular expression?

Challenges Involving Regular Expressions

1. Is there a simple procedure to test if a given string satisfies a regular expression?

$$11110010 \in \{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*) \qquad ???$$

Challenges Involving Regular Expressions

1. Is there a simple procedure to test if a given string satisfies a regular expression?

$$11110010 \in \{1, 11\} \bullet \overline{\{0, 01\}^*} \bullet (\{00\} \cup \{1\}^*) \qquad ???$$

2. Regular expression for all palindromes (strings which equal their reversal)?

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**

2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow \begin{array}{l} 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}, \\ 1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}. \end{array}$ **[constructor rules]**

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**

2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**

3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$

Give recursive definitions of these languages.

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$

Give recursive definitions of these languages.

Give regular expressions for these languages.

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$
Give recursive definitions of these languages.
Give regular expressions for these languages.

These computing problems look similar.

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$
Give recursive definitions of these languages.
Give regular expressions for these languages.

These computing problems look similar.

They are **VERY** different. Which do you think is more "complex"?

Recursively Defined Languages: Palindromes

1. $\varepsilon, 0, 1 \in \mathcal{L}_{\text{palindrome}}$. **[basis]**
2. $w \in \mathcal{L}_{\text{palindrome}} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{\text{palindrome}}$, $1 \bullet w \bullet 1 \in \mathcal{L}_{\text{palindrome}}$. **[constructor rules]**
3. Nothing else is in $\mathcal{L}_{\text{palindrome}}$. **[minimality]**

Pop Quiz. Similar looking languages: $\{0^{\bullet n}1^{\bullet k} \mid n, k \geq 0\}$ and $\{0^{\bullet n}1^{\bullet n} \mid n \geq 0\}$
Give recursive definitions of these languages.
Give regular expressions for these languages.

These computing problems look similar.

They are **VERY** different. Which do you think is more "complex"?

How to define complexity of a computing problem?

Complexity of a Computing Problem

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership?

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

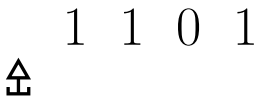
How do we test membership? That brings us to *Models Of Computing*.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.



Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" YES-set $\leftrightarrow$ hard to test membership in YES-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" $\boxed{\text{YES}}$-set $\leftrightarrow$ hard to test membership in $\boxed{\text{YES}}$-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.
3: In state q_1, when you process a 0, transition to state q_0.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" YES-set $\leftrightarrow$ hard to test membership in YES-set

How do we test membership? That brings us to *Models Of Computing*.

Visual encoding of four (machine-level) instructions:

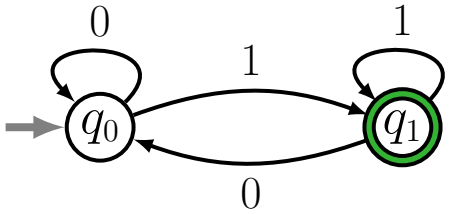
1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.
3: In state q_1, when you process a 0, transition to state q_0.
4: In state q_1, when you process a 1, transition to state q_1.

Complexity of a Computing Problem

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, 1011, \ldots\}$ (strings ending in 1)

difficult problem $\leftrightarrow$ "complex" YES-set $\leftrightarrow$ hard to test membership in YES-set

How do we test membership? That brings us to *Models Of Computing*.



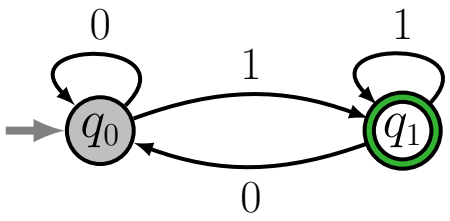
Visual encoding of four (machine-level) instructions:

1: In state q_0, when you process a 0, transition to state q_0.
2: In state q_0, when you process a 1, transition to state q_1.
3: In state q_1, when you process a 0, transition to state q_0.
4: In state q_1, when you process a 1, transition to state q_1.

"Easy" to implement as a mechanical device.

A Simple Computing Machine (DFA)

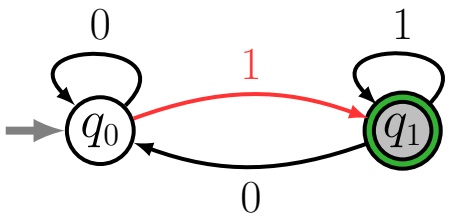
1 1 0 1



(current state in gray)

A Simple Computing Machine (DFA)

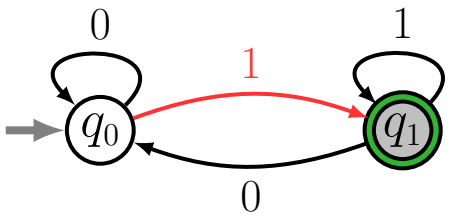
1 1 0 1



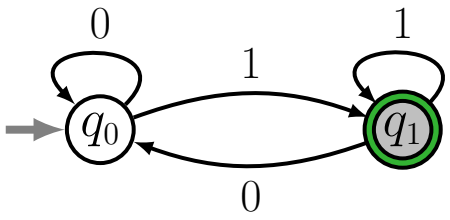
(current state in gray)

A Simple Computing Machine (DFA)

1 1 0 1



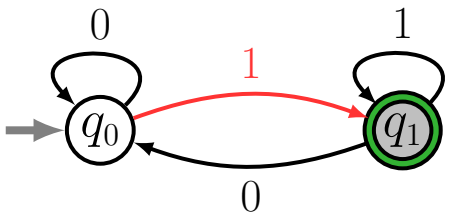
1 1 0 1



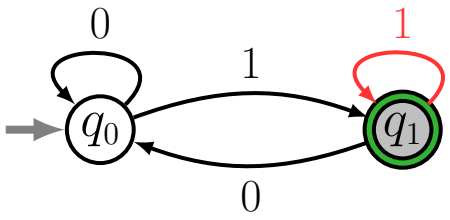
(current state in gray)

A Simple Computing Machine (DFA)

1 1 0 1



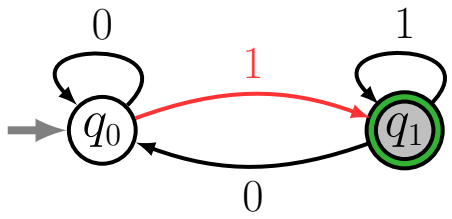
1 1 0 1



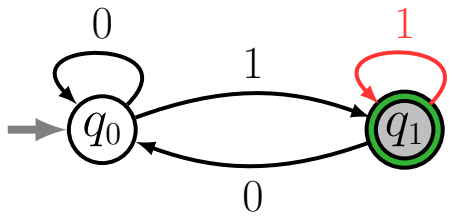
(current state in gray)

A Simple Computing Machine (DFA)

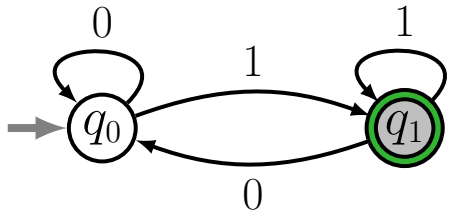
1 1 0 1



1 1 0 1



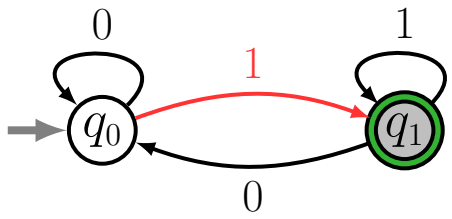
1 1 0 1



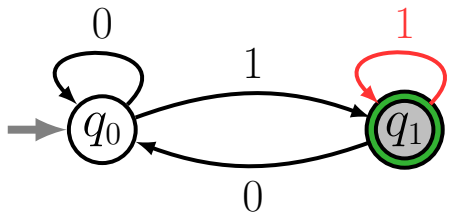
(current state in gray)

A Simple Computing Machine (DFA)

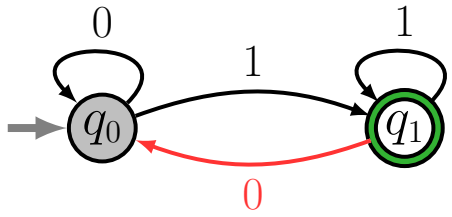
1 1 0 1



1 1 0 1



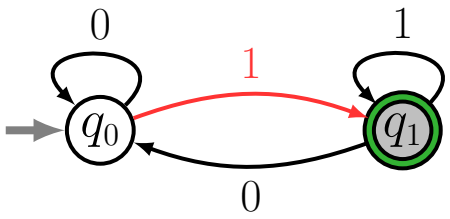
1 1 0 1



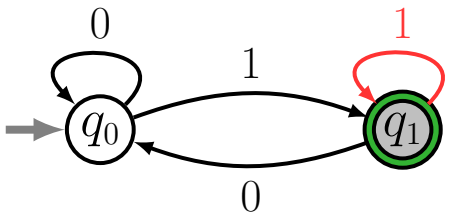
(current state in gray)

A Simple Computing Machine (DFA)

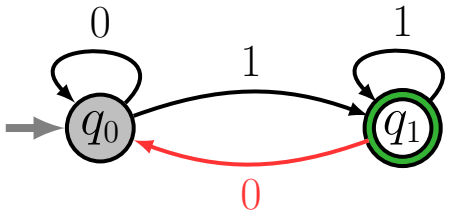
1 1 0 1



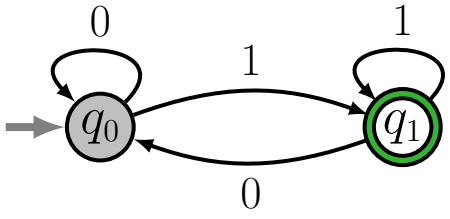
1 1 0 1



1 1 0 1



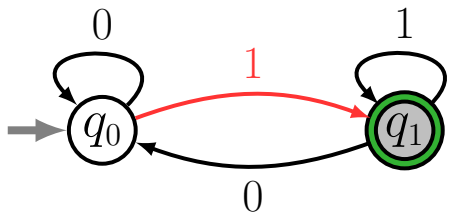
1 1 0 1



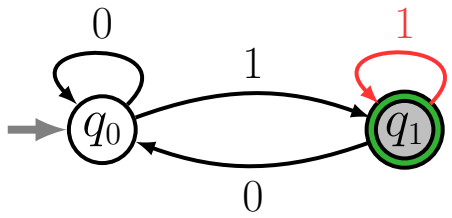
(current state in gray)

A Simple Computing Machine (DFA)

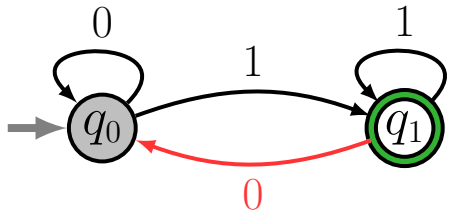
1 1 0 1



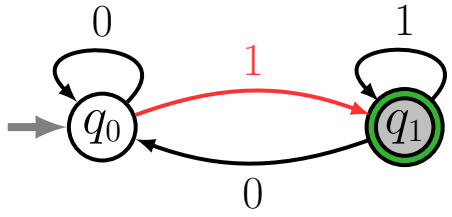
1 1 0 1



1 1 0 1



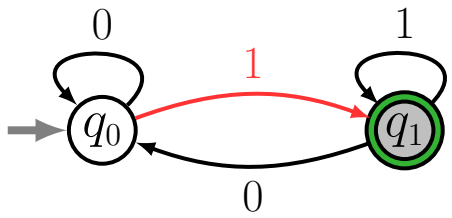
1 1 0 1



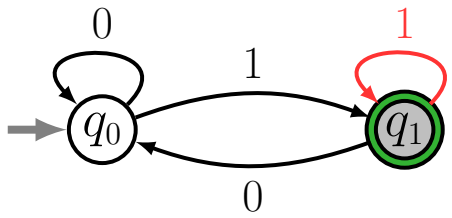
(current state in gray)

A Simple Computing Machine (DFA)

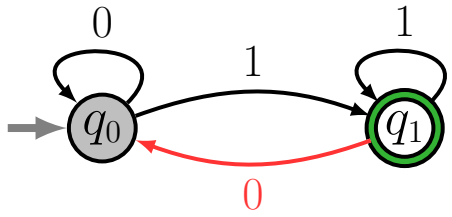
1 1 0 1



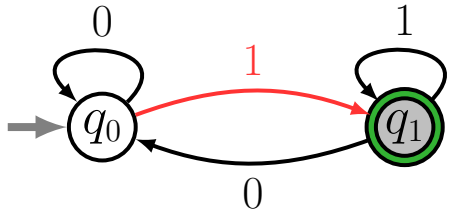
1 1 0 1



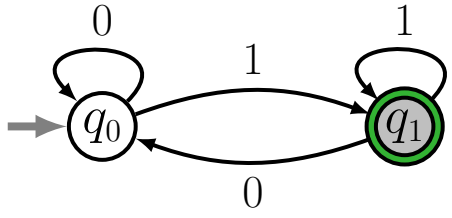
1 1 0 1



1 1 0 1



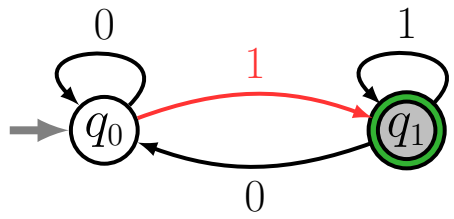
1 1 0 1



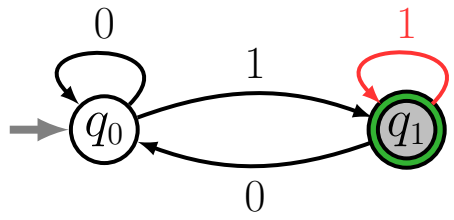
(current state in gray)

A Simple Computing Machine (DFA)

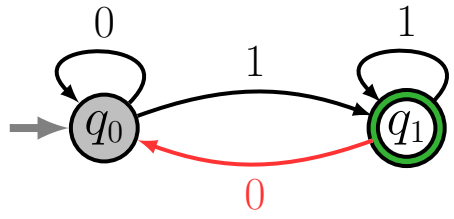
1 1 0 1



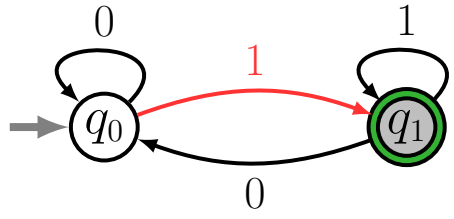
1 1 0 1



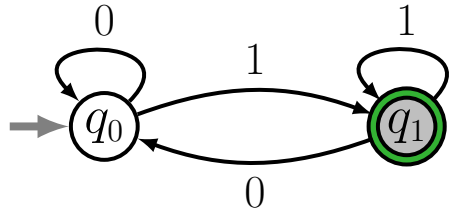
1 1 0 1



1 1 0 1



1 1 0 1



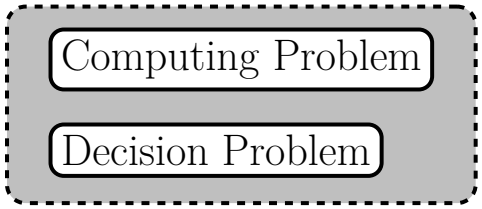
(current state in gray)

$\mathcal{L}_{\text{push}} = \{1, 01, 11, 001, 011, 101, 111, 0001, \ldots\}$

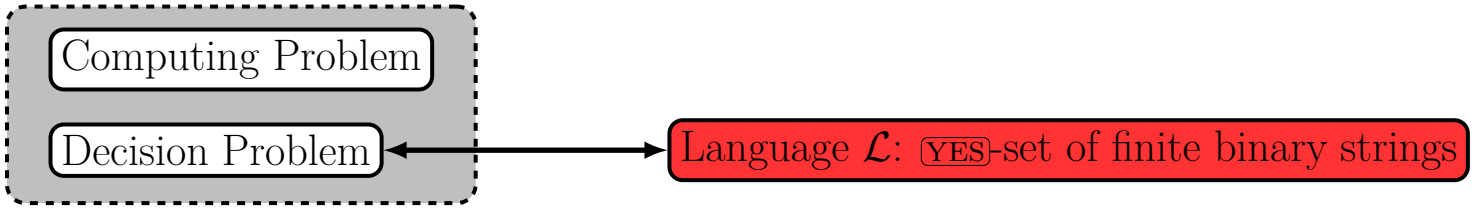
Strings in $\mathcal{L}_{\text{push}}$ end in the "accepting" state q_1.

Strings not in $\mathcal{L}_{\text{push}}$ do not.

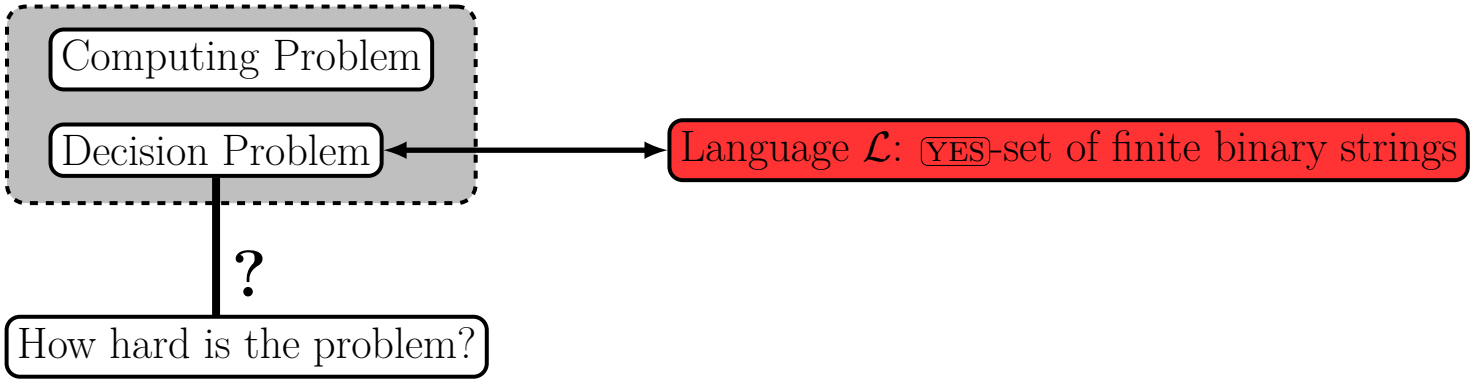
Computing Problems and Their Difficulty



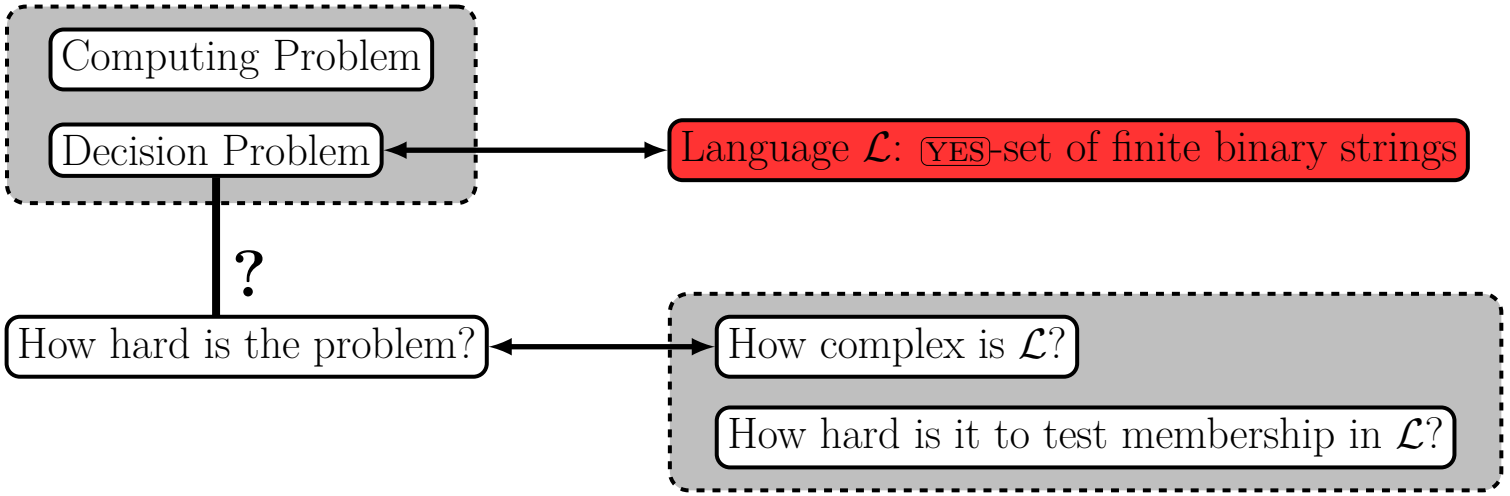
Computing Problems and Their Difficulty



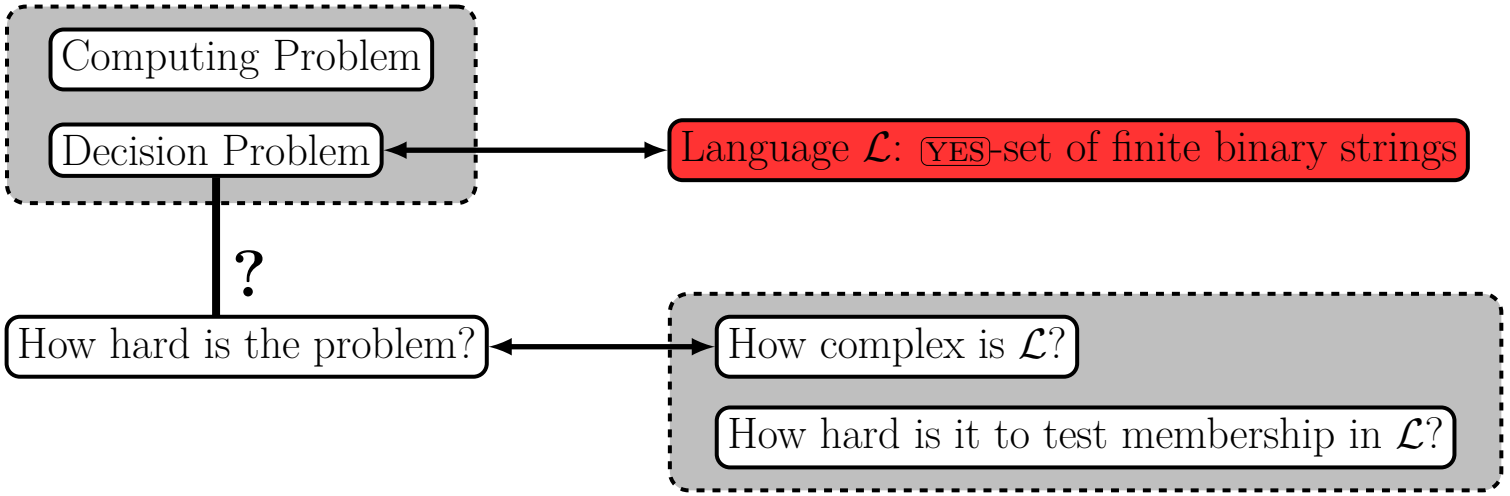
Computing Problems and Their Difficulty



Computing Problems and Their Difficulty



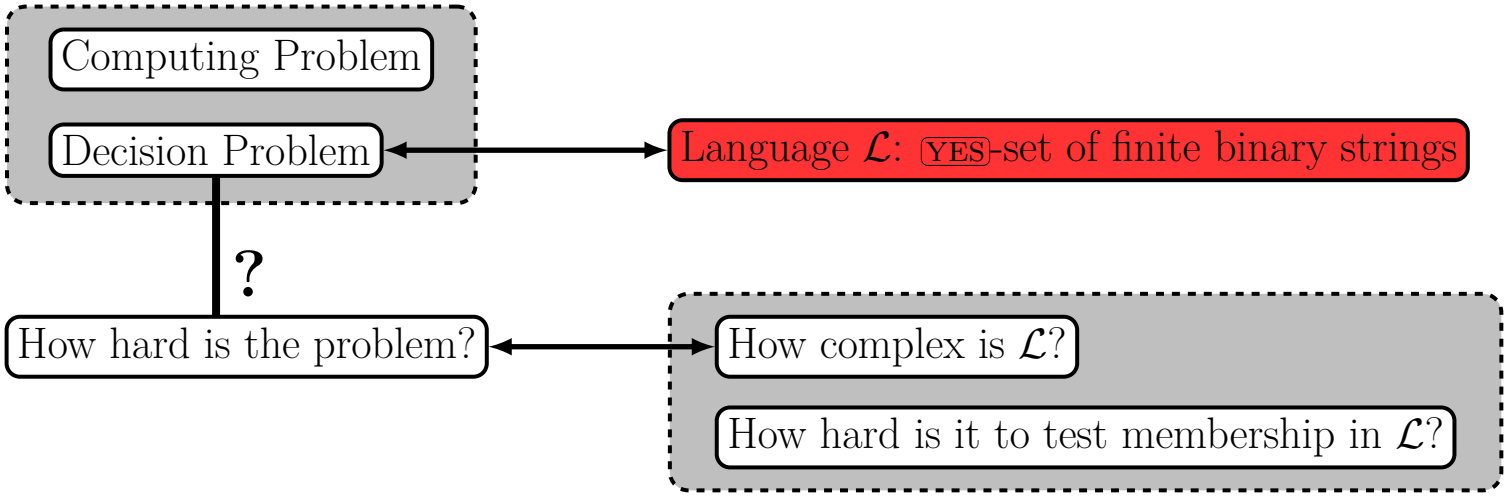
Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.

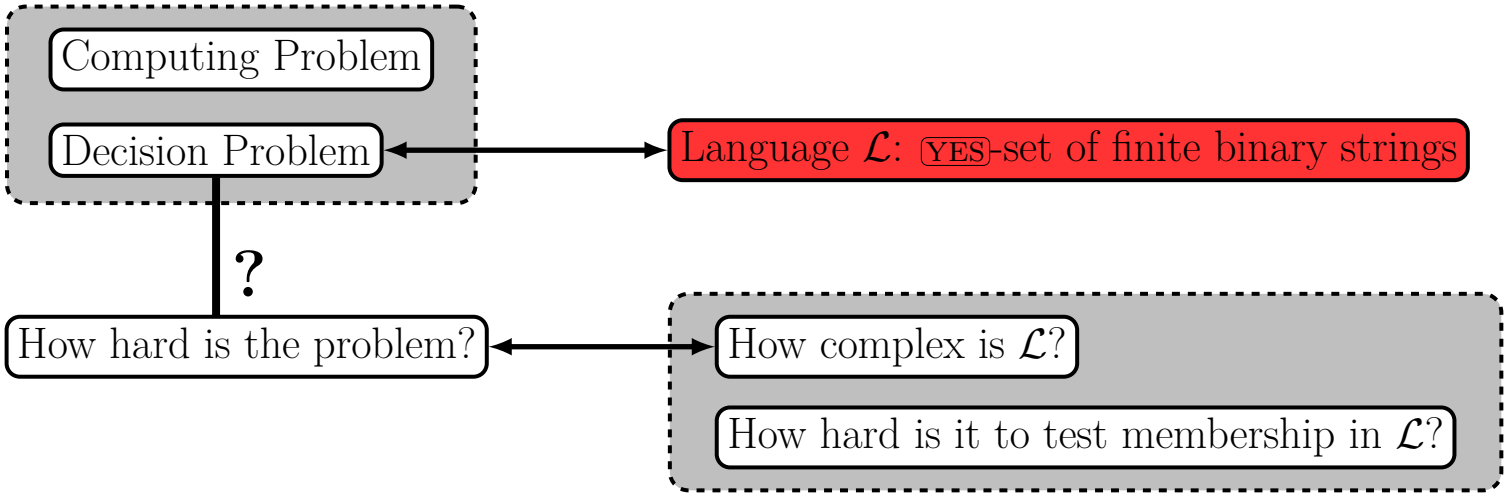
Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.
2. The problem needs a different *kind* of computing machine, with superior capabilities.

Computing Problems and Their Difficulty



A problem can be harder in two ways.

1. The problem needs more resources. For example, the problem can be solved with a similar machine to ours, except with more states.
2. The problem needs a different *kind* of computing machine, with superior capabilities.

The first type of "harder" is the focus of a follow-on algorithms course.

We focus on what *can and can't be solved* on a particular kind of machine.