Foundations of Computer Science Lecture 4

Proofs

Proving "If ... then ..." (Implication): Direct proof; Contraposition
Contradiction
Proofs About Sets

Last Time

1. How to make precise statements.
2. Quantifiers which allow us to make statements about many things.

Today: Proofs

1. Proving "IF ..., THEN ...".

2. Proof Patterns
   - Direct Proof

Implications: Reasoning in the Absence of Facts

Reasoning:

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IF we can quickly find the largest friend-clique in a friendship network,

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More Mathematical Example: Quadratic formula.

IF $ax^2 + bx + c = 0$ and $a \neq 0$, THEN $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.

Proving an Implication

$$\text{IF } \underbrace{x \text{ and } y \text{ are rational}}_{p}, \text{ THEN } \underbrace{x + y \text{ is rational}}_{q}.$$

$$\forall (x, y) \in \mathbb{Q}^2 : \underbrace{x + y \text{ is rational}}_{P(x,y)}.$$

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Proof. We must show that the row $p = \text{T}$, $q = \text{F}$ can't happen.

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F	F	T
F	T	T
T	F	F
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$x = \frac{a}{b}$ and $y = \frac{c}{d}$, where $a, c \in \mathbb{Z}$ and $b, d \in \mathbb{N}$.

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That means q is T.

The row $p = \text{T}$, $q = \text{F}$ *cannot* occur and the implication is proved. ■

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Theorem. If $x, y \in \mathbb{Q}$, then $x + y \in \mathbb{Q}$.

Proof. We prove the theorem using a direct proof.

1: Assume that $x, y \in \mathbb{Q}$, that is x and y are rational.
2: Then there are integers a, c and natural numbers b, d such that $x = a/b$ and $y = c/d$ (because this is what it means for x and y to be rational).
3: Then $x + y = (ad + bc)/bd$ (high-school algebra).
4: Since $ad + bc \in \mathbb{Z}$ and $bd \in \mathbb{N}$, $(ad + bc)/bd$ is rational.
5: Thus, we conclude (from steps 3 and 4) that $x + y \in \mathbb{Q}$. ■

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(VI) **Read your proof.** Finally, check correctness; edit; simplify.

Example: Direct Proof

Let x be any real number, i.e. $x \in \mathbb{R}$.

$$\textsc{if}\ \underbrace{4^x - 1 \text{ is divisible by } 3}_{p},\ \textsc{then}\ \underbrace{4^{x+1} - 1 \text{ is divisible by } 3}_{q}.$$

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5: Therefore, the statement claimed in q is T. ■

Question. Is $4^x - 1$ divisible by 3?

We Made No Assumptions About x

$$P(x) : \text{“IF } 4^x - 1 \text{ is divisible by 3, THEN } 4^{x+1} - 1 \text{ is divisible by 3”}$$

Since we made no assumptions about x, we proved:

$$\forall x \in \mathbb{R} : P(x)$$

Exercise. Prove: For all pairs of odd integers m, n, the sum $m + n$ is an even integer.

Practice. Exercise 4.2.

Disproving an Implication

$$\text{IF } \underbrace{x^2 > y^2}_{p}, \text{ THEN } \underbrace{x > y}_{q}.$$

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Counter-example: $x = -8, y = -4$.

$x^2 > y^2$ so, $p = \text{T}$

$x < y$ so, $q = \text{F}$

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The row $p = \text{T}, q = \text{F}$ has occurred!

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A single **counter-example** suffices to disprove an implication.

Contraposition

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x is odd, $x = 2k + 1$.

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$$\begin{aligned} x^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \;=\; 2(2k^2 + 2k) + 1 \qquad \leftarrow \text{ odd} \end{aligned}$$

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That means p is F.

The row $p = \text{T}$, $q = \text{F}$ *cannot* occur!

The implication is proved. ■

Template: Contraposition Proof of an Implication $p \to q$

Proof. We prove the theorem using contraposition.

1: Start by assuming that the statement claimed in q is F.
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Theorem. If x^2 is even, then x is even.

Proof. We prove the theorem by contraposition.

1: Assume that x is odd.
2: Then $x = 2k + 1$ for some $k \in \mathbb{Z}$ (that's what it means for x to be odd)
3: Then $x^2 = 2(2k^2 + 2k) + 1$ (high-school algebra).
4: Which means x^2 is 1 plus a multiple of 2, and hence is odd.
5: We have shown that x^2 is odd, concluding the proof. ■

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Exercise. Prove: IF r is irrational, THEN $\sqrt{r}$ is irrational.

Equivalence: . . . IF AND ONLY IF. . .

p and q are equivalent means they are either both T or both F.

p IF AND ONLY IF q or $p \leftrightarrow q$

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F	T	F
T	F	F
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To prove $p \leftrightarrow q$ is T, you must prove:

1. Row $p = T, q = F$ cannot occur: that is $p \rightarrow q$.
2. Row $p = F, q = T$ cannot occur: that is $q \rightarrow p$.

Integer x is divisible by 3 IF AND ONLY IF x^2 is divisible by 3.

$$\underbrace{x \text{ is divisible by } 3}_{p} \text{ IF AND ONLY IF } \underbrace{x^2 \text{ is divisible by } 3}_{q}.$$

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(i) **Prove $p \to q$: if x is divisible by 3, then x^2 is divisible by 3.**

(ii) **Prove $q \to p$: if x^2 is divisible by 3, then x is divisible by 3.**

■

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Proof. The proof has two main steps (one for each implication):

(i) **Prove $p \to q$: if x is divisible by 3, then x^2 is divisible by 3.**
We use a direct proof. Assume x is divisible by 3, so $x = 3k$ for some $k \in \mathbb{Z}$.
Then, $x^2 = 9k^2 = 3 \cdot (3k^2)$ is a multiple of 3, and so x^2 is divisible by 3.

(ii) **Prove $q \to p$: if x^2 is divisible by 3, then x is divisible by 3.**

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We use contraposition. Assume x is not divisible by 3. There are two cases for x,

Case 1: $x = 3k + 1 \;\to\; x^2 = 3k(3k + 2) + 1$ (1 more than a multiple of 3).

Case 2: $x = 3k + 2 \;\to\; x^2 = 3(3k^2 + 4k + 1) + 1$ (1 more than a multiple of 3).

In all cases, x^2 is not divisible by 3, as was to be shown. ■

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- IF AND ONLY IF proof contains the proofs of *two* implications.
- Each implication may be proved differently.

Contradictions

$$1 = 2; \qquad n^2 < n \text{ (for integer } n); \qquad |x| < x; \qquad p \wedge \neg p.$$

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1: **Assume $\sqrt{2}$ is rational.**
2: This means $\sqrt{2} = a_*/b_*$; b_* is the smallest denominator (well-ordering).
3: That is, a_* and b_* cannot have 2 as a common factor.
4: We have: $2 = a_*^2/b_*^2 \to a_*^2 = 2b_*^2$, or a_*^2 is even. Hence, a_* is even, $a_* = 2k$. [we proved this]
5: Therefore, $4k^2 = 2b_*^2$ and so $b_*^2 = 2k^2$, or b_*^2 is even. Hence, b_* is even, $b_* = 2\ell$.
6: Hence, a_* and b_* are both divisible by 2. **(FISHY)**

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What could possibly be wrong with this derivation? It must be step 1.

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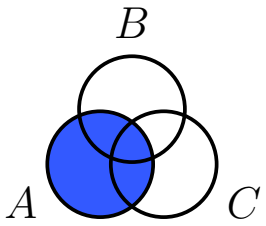
Exercise. Let a, b be integers. Prove that $a^2 - 4b \neq 2$.

Proofs about Sets

Venn diagram proofs: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.



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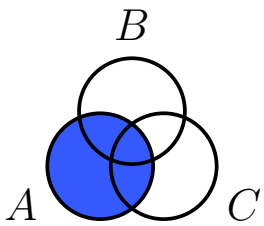
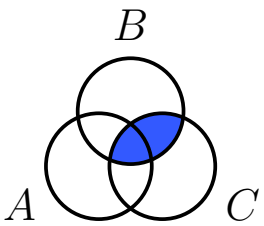


A



$B \cap C$

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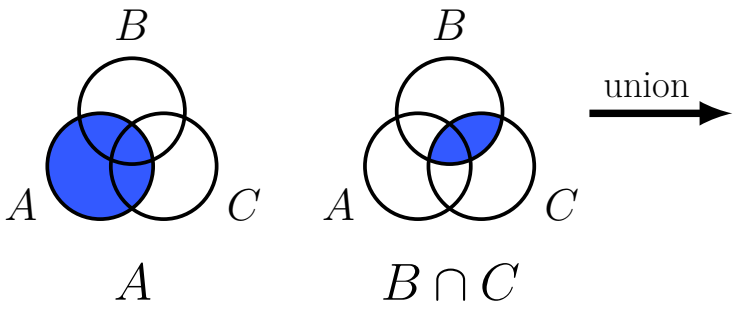

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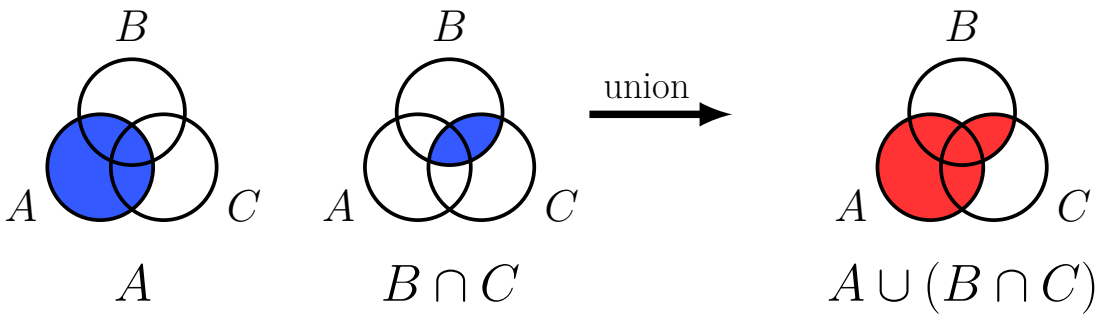

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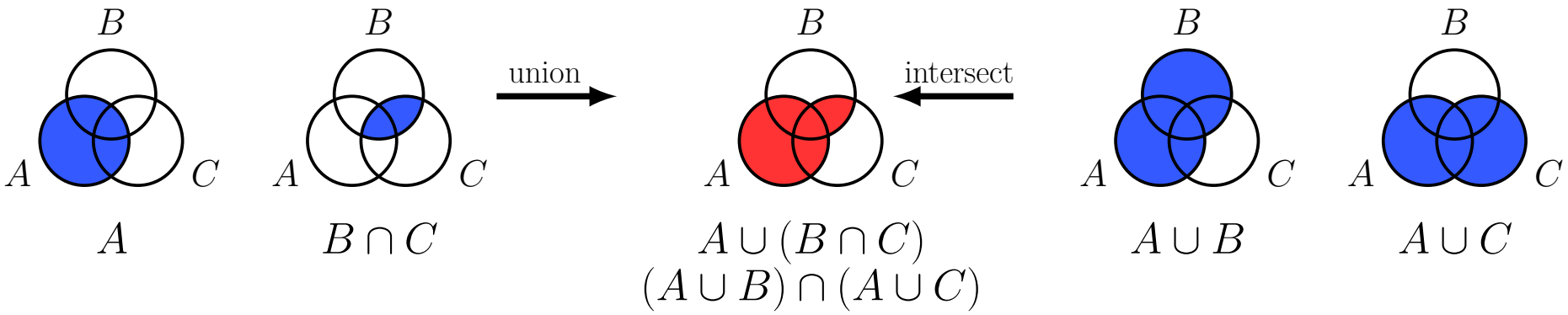

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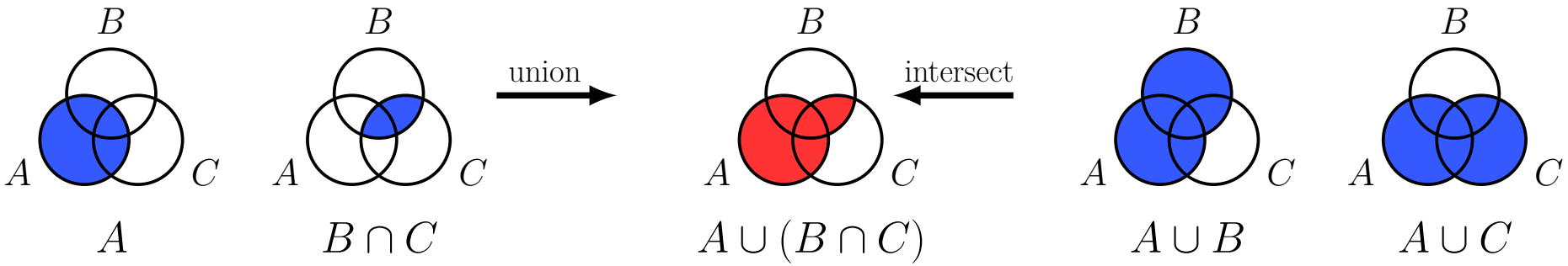

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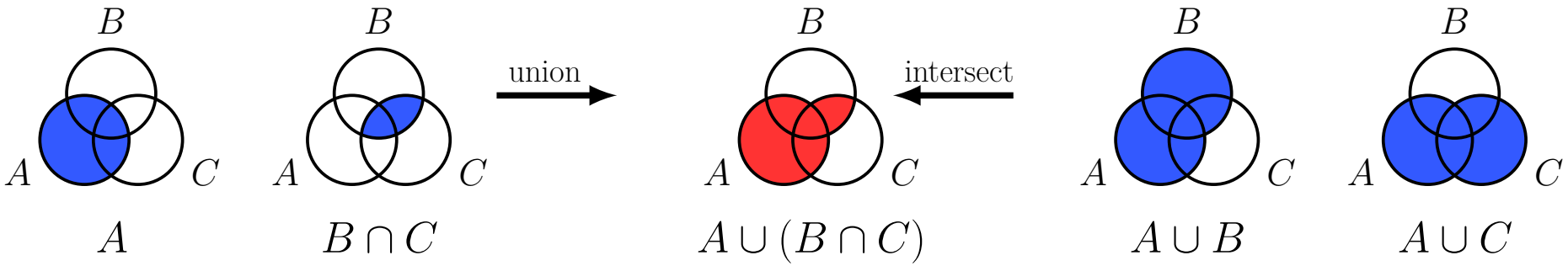

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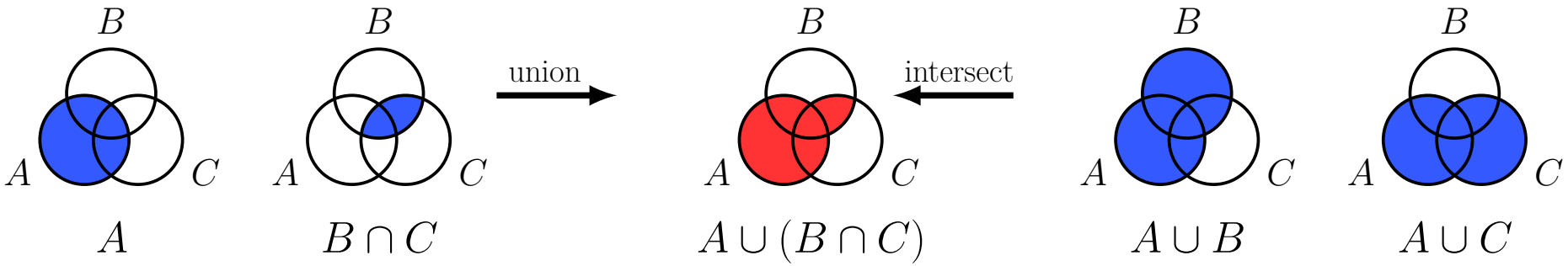

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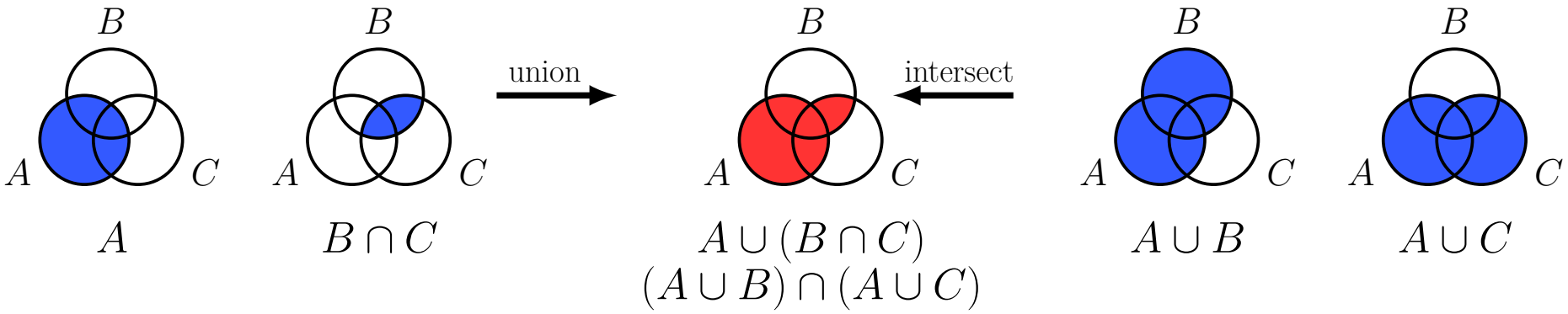

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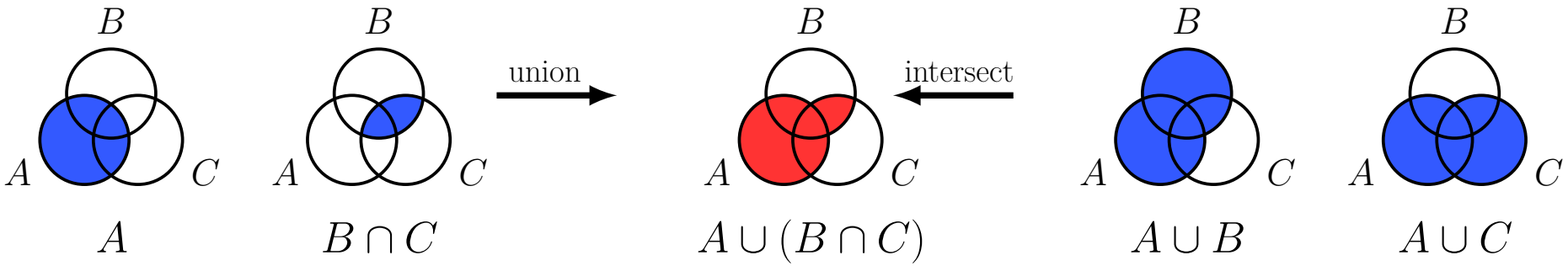

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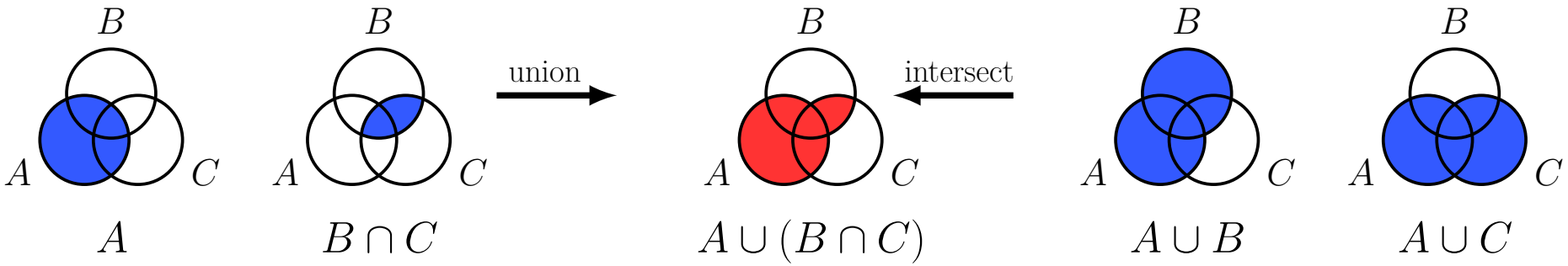

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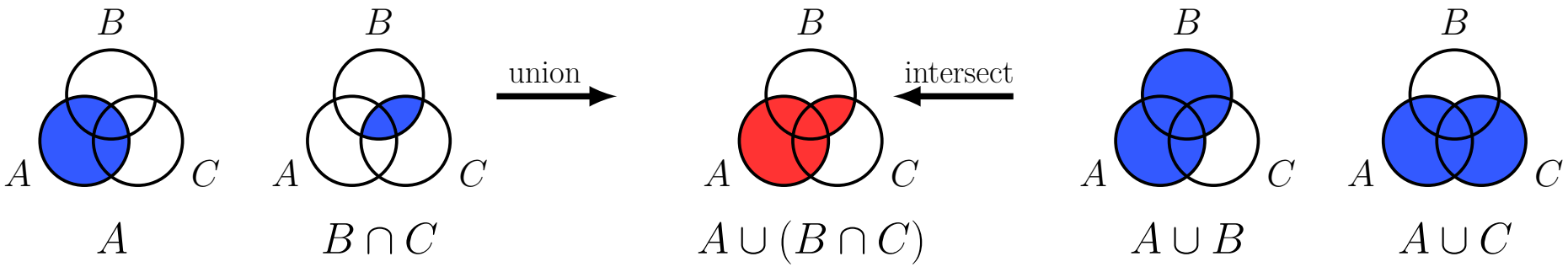

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Exercise. $A = \{\text{multiples of } 2\}$; $B = \{\text{multiples of } 9\}$; $C = \{\text{multiples of } 6\}$. Prove $A \cap B \subseteq C$.

Picking a Proof Template

Situation you are faced with	Suggested proof method

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	**Contraposition**

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	**Show an example**

Picking a Proof Template

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2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	**Show an example**
4. Prove something does not exist	

Picking a Proof Template

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2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**

Picking a Proof Template

Situation you are faced with	Suggested proof method
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3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**
5. Prove something is unique	

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**
5. Prove something is unique	**Contradiction**

Picking a Proof Template

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3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**
5. Prove something is unique	**Contradiction**
6. Prove something is *not true* for *all* objects	

Picking a Proof Template

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3. Prove something exists	**Show an example**
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6. Prove something is *not true* for *all* objects	**Show a counter-example**

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3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**
5. Prove something is unique	**Contradiction**
6. Prove something is *not true* for *all* objects	**Show a counter-example**
7. Show something is *true* for *all* objects	

Picking a Proof Template

Situation you are faced with	Suggested proof method
1. Clear how result follows from assumption	**Direct proof**
2. Clear that if result is false, the assumption is false	**Contraposition**
3. Prove something exists	**Show an example**
4. Prove something does not exist	**Contradiction**
5. Prove something is unique	**Contradiction**
6. Prove something is *not true* for *all* objects	**Show a counter-example**
7. Show something is *true* for *all* objects	**Show for general object**

Practice. Exercise 4.8.