Machine Learning and Optimization

Lecture 8: Subdifferentials and Nonsmooth Convex Optimization

CSCI 4140/6960 — Spring 2026

Course website: `cs.rpi.edu/~gittea/.../mlandopt.html`

Learning Objectives

By the end of this lecture, you should be able to:

1. Define extended-value convex functions, proper functions, and lower semicontinuous (lsc) functions, and explain why lsc is essential for optimization
2. Reformulate constrained convex problems as unconstrained problems using indicator functions
3. Define and compute subdifferentials for simple nonsmooth convex functions

Outline

Beyond Smooth Optimization

Extended-Value Functions and Lower Semicontinuity

Subdifferentials: Definition and Geometric Intuition

Computing Subdifferentials: Simple Examples

Summary and Looking Ahead

The Problem with Gradient Methods

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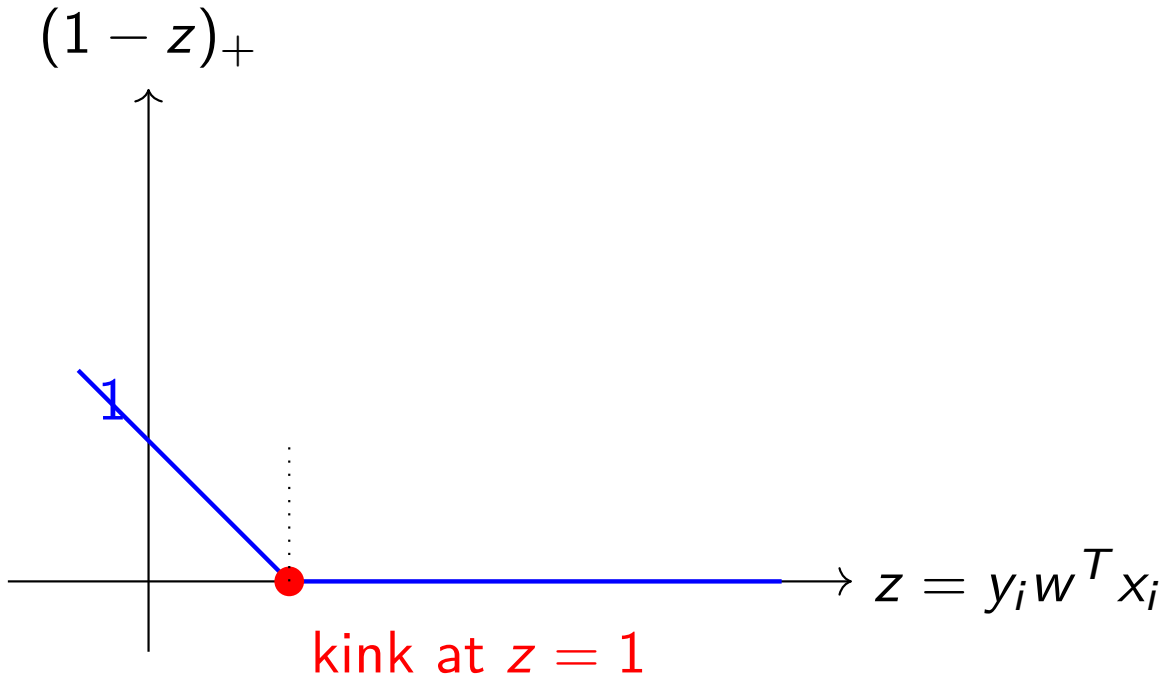
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- $f(x) = |x|$ (absolute value)
- $f(x) = \|x\|_1$ (ℓ_1 regularization / LASSO)
- $f(w) = \frac{1}{n}\sum_{i=1}^{n}(1 - y_i w^T x_i)_+$ (SVM objective)

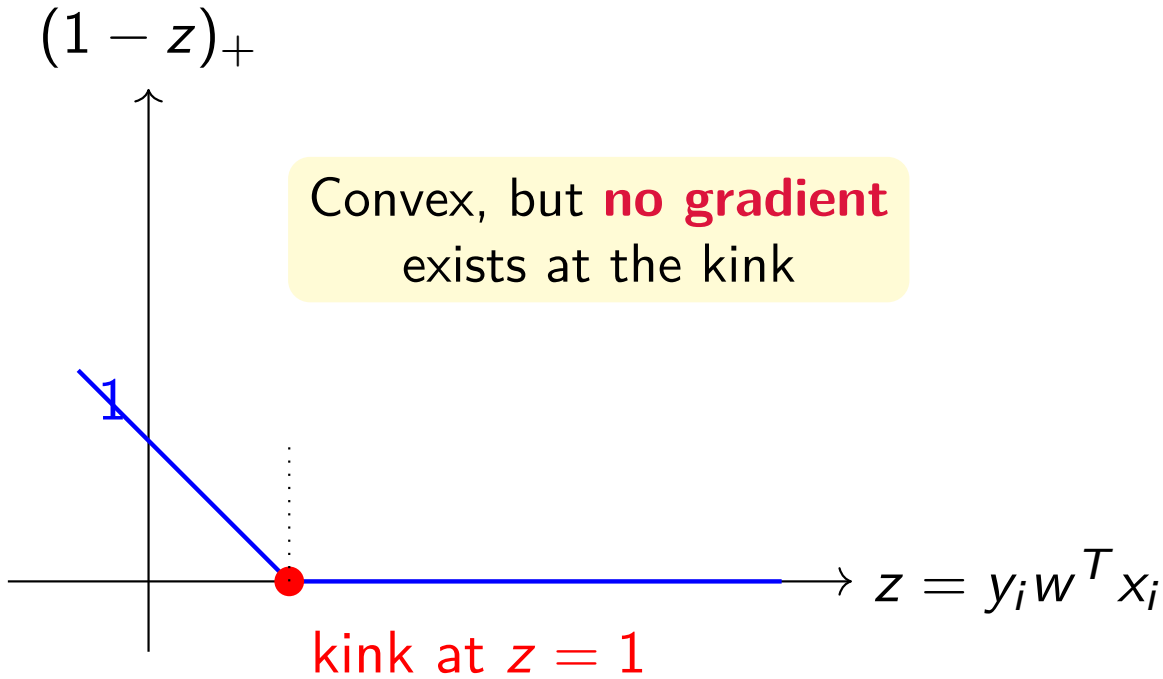
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Piecewise Linear Structure of the Hinge Loss

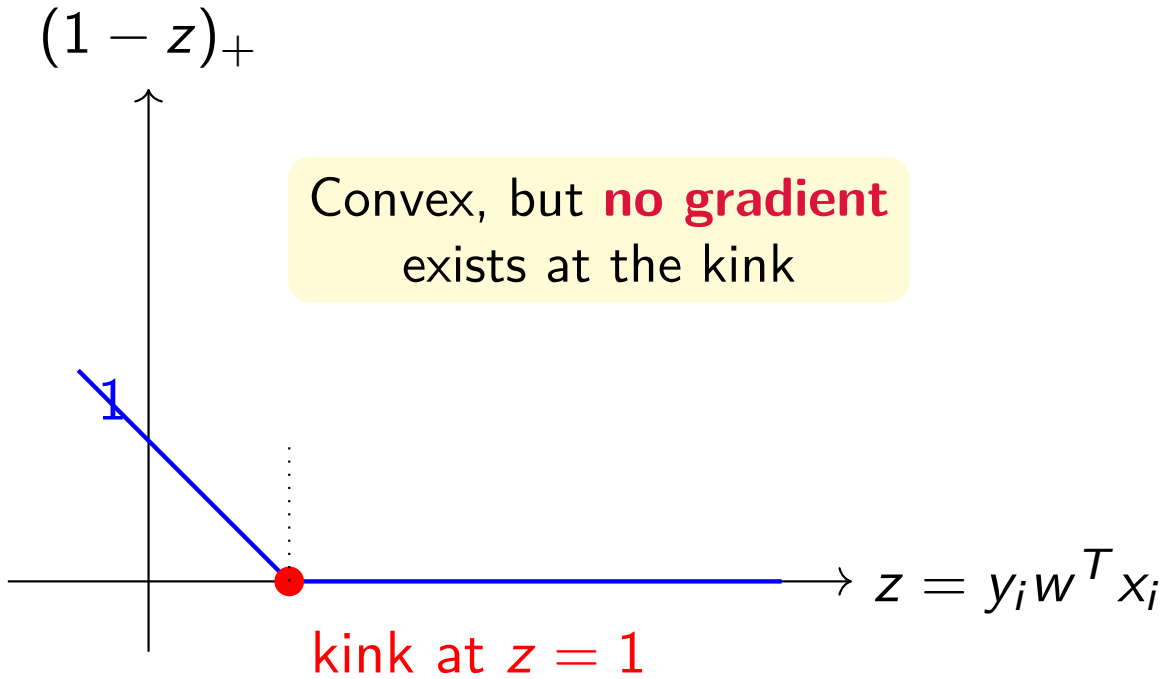


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Answer: use optimality conditions and algorithms developed for nondifferentiable convex functions!

What replaces differentiability?

When we drop differentiability, we lose several intuitions:

- **Existence:** Does a minimizer even exist? (Not always!)
- **Boundaries:** What happens at the "edge" of the function's domain?
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To build robust theory and algorithms, we need appropriate assumptions:

1. **Extended-value functions** (∞) to encode constraints as nondifferentiable objectives.
2. **Properness** to avoid triviality.
3. **Lower semicontinuity** to ensure objectives behave at the limit.

Extended-Value and Proper Functions

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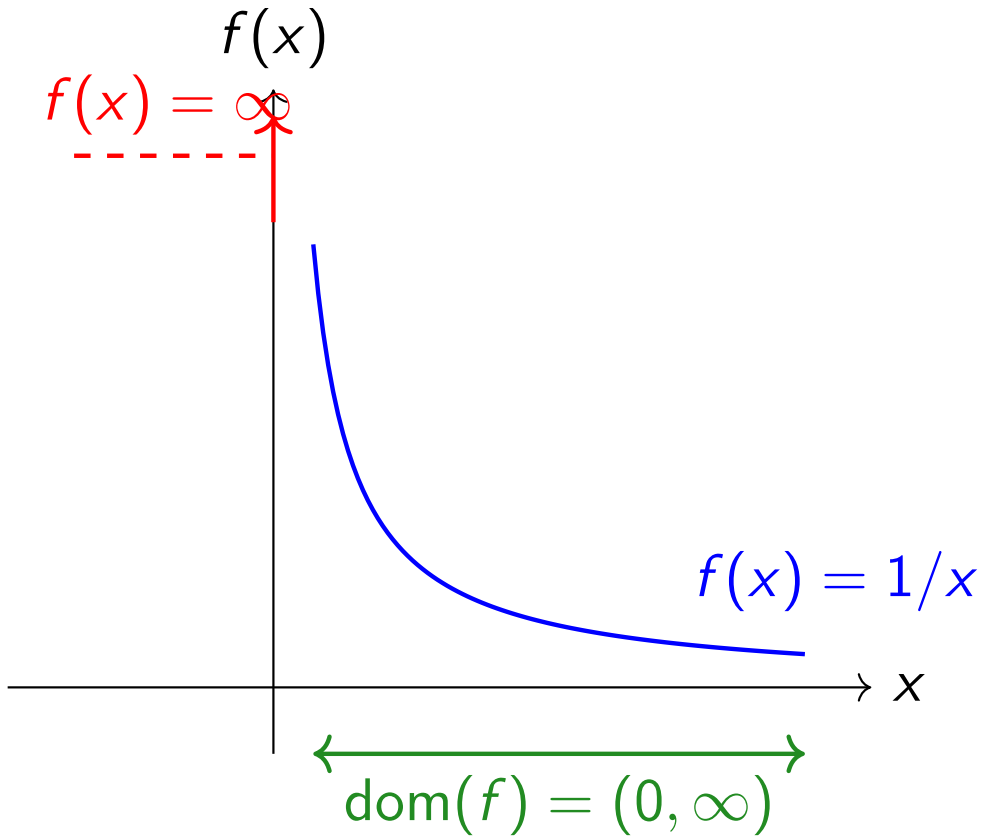
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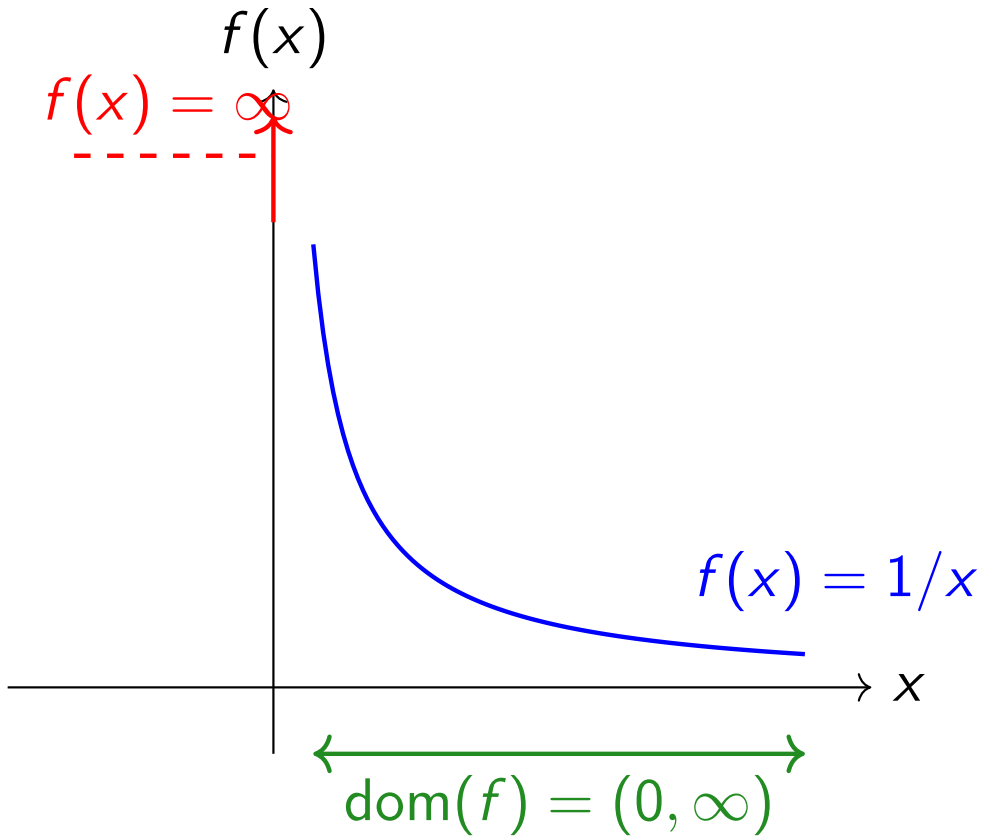
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Why allow ∞? This lets us encode constraints implicitly, unifying constrained and unconstrained optimization.

Example: Extended-Value Function



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$$f(x) = \begin{cases} 1/x & x > 0 \\ \infty & x \leq 0 \end{cases}$$ is a proper extended-value convex function.

Lower Semicontinuity (lsc)

Definition

$f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is **lower semicontinuous (lsc)** at x if:

$$\liminf_{y \to x} f(y) \geq f(x)$$

Equivalently, for every sequence $x_n \to x$: $f(x) \leq \liminf_{n \to \infty} f(x_n)$.
Equivalently, the epigraph of f is closed (hence you will sometimes hear such functions referred to as "closed").

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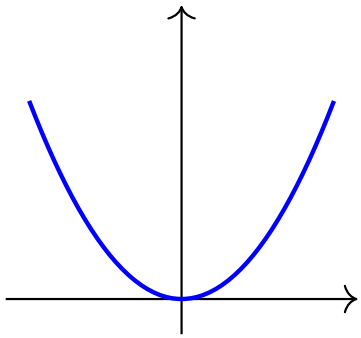
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- Downward jumps are allowed (lsc)
- Upward jumps are **forbidden** (breaks lsc)

Visualizing Lower Semicontinuity

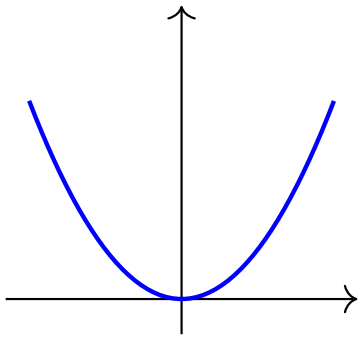


Continuous $\Rightarrow$ lsc

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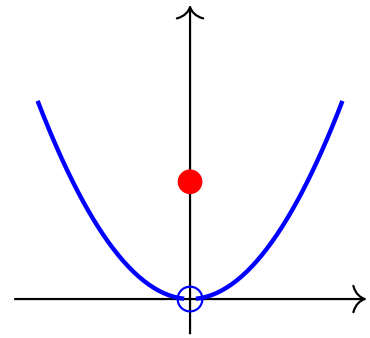
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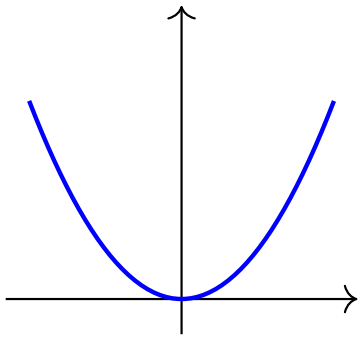


Jumps **up**: **not lsc**

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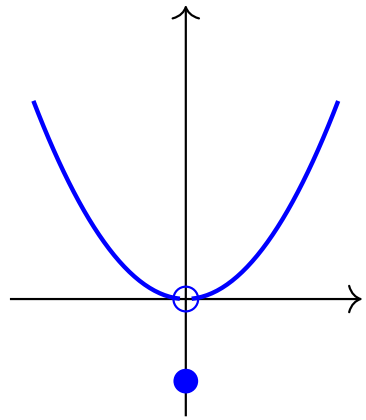
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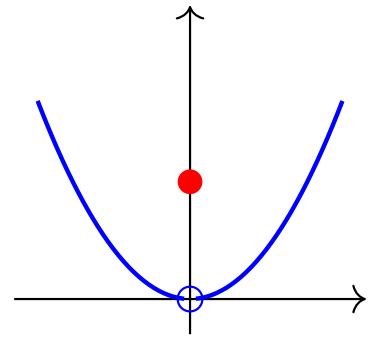
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Jumps down: lsc

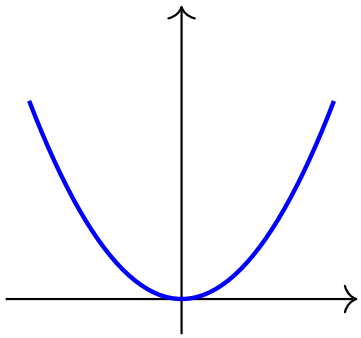
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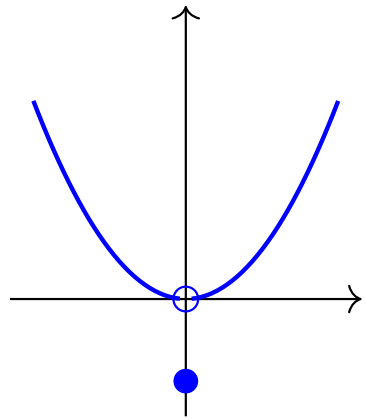
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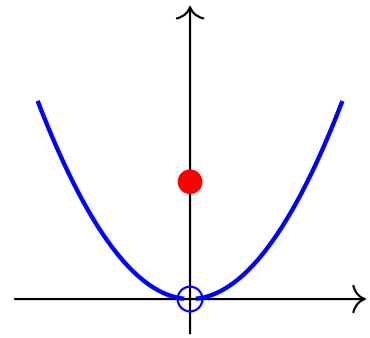
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Why LSC Matters: Existence of Minimizers

Theorem (Existence of Minimizers)

If $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ *is proper, lsc, and coercive* ($f(x) \to \infty$ *as* $\|x\| \to \infty$)*, then* f *attains its minimum.*

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Why these assumptions are exactly what you need:

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Why compactness $\Rightarrow$ minimum is attained (proof sketch):
Let $L_c = \{x : f(x) \leq c\}$ and $m = \inf_{x \in L_c} f(x)$.

- Pick a minimizing sequence $x_n \in L_c$ with $f(x_n) \downarrow m$
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If *any* assumption fails, the infimum may exist but be unattainable.

Proof Sketch: Compactness $\Rightarrow$ Minimum is Attained

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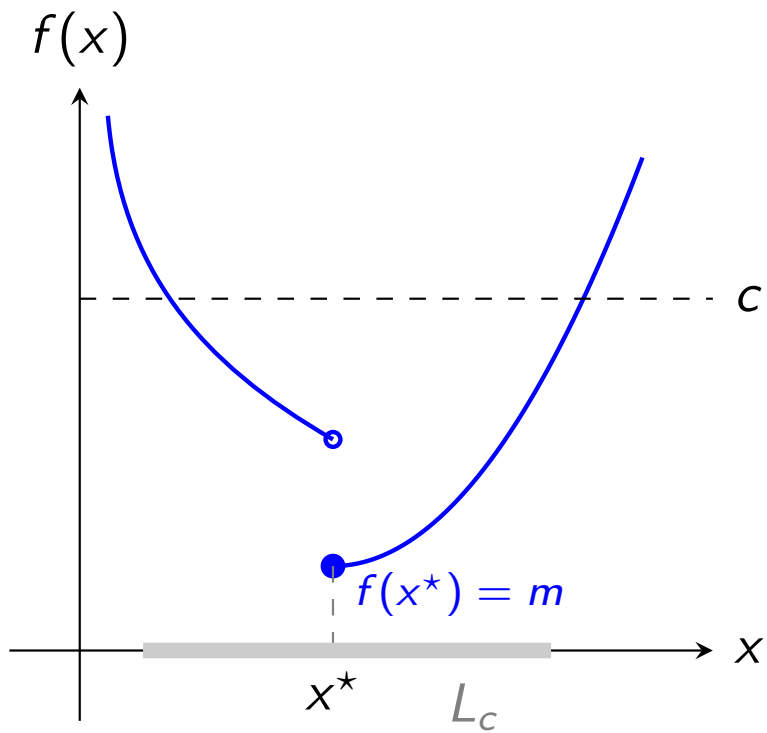
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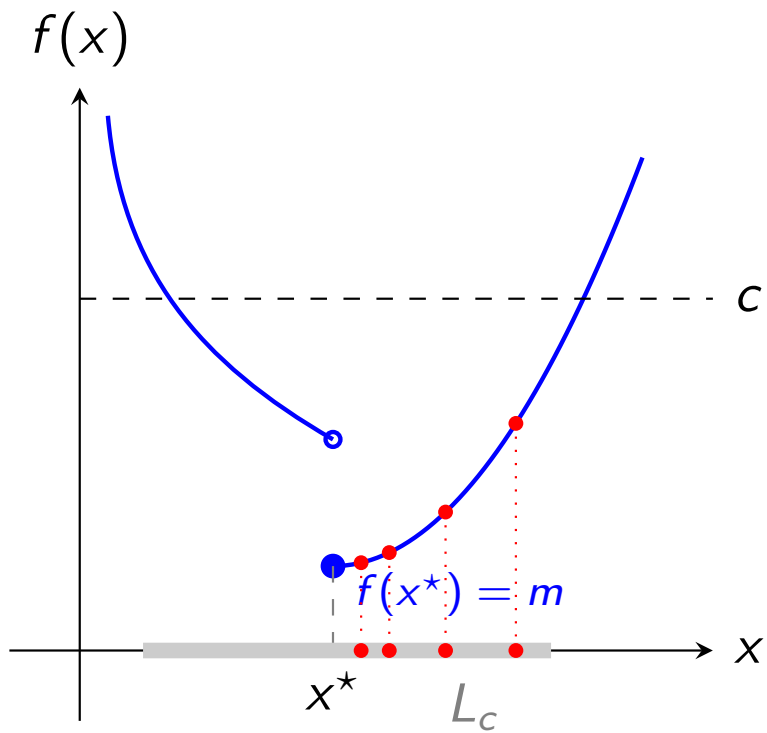
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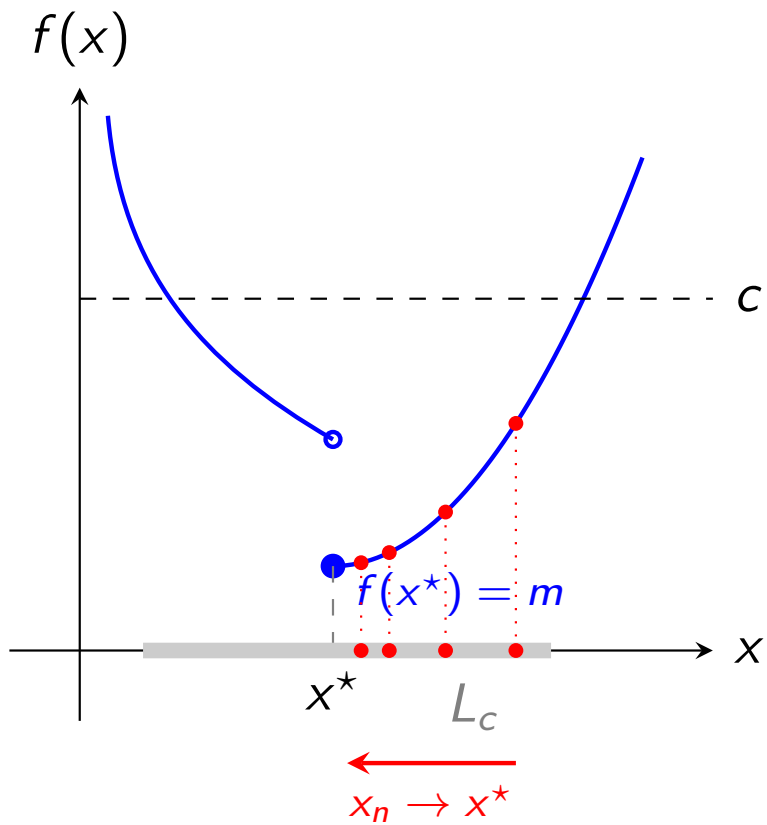
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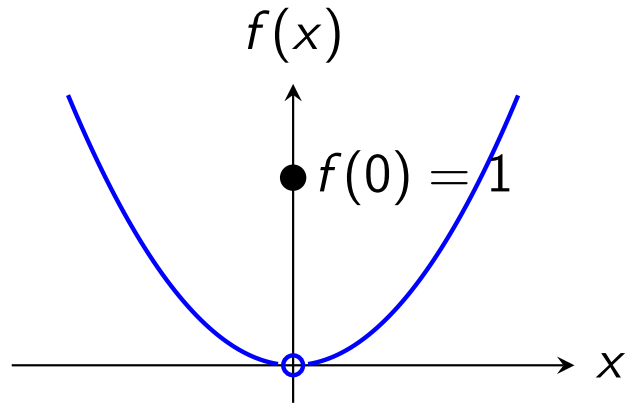
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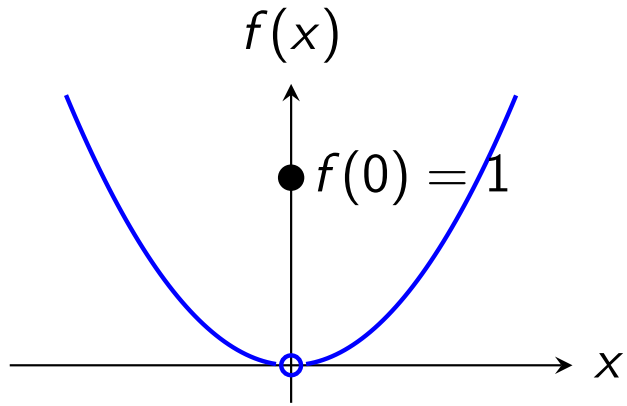


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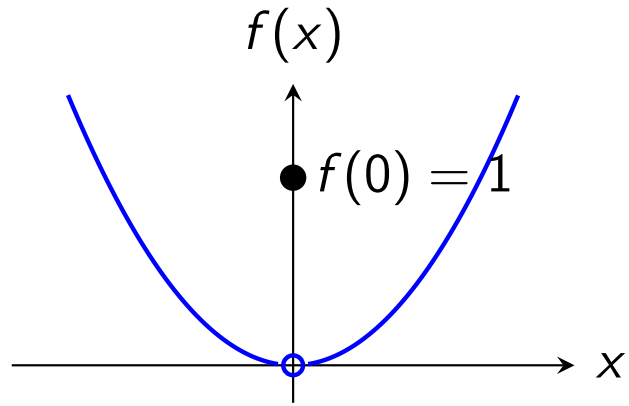
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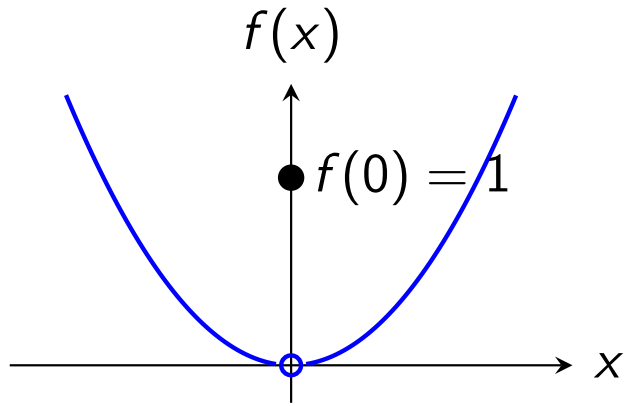


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- $\inf_x f(x) = 0$ (as $x \to 0$)
- But $f(x) > 0$ for all x
- **No minimizer exists!**
- Not lsc at $x = 0$

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The Convex Indicator Function

For a convex set $C \subseteq \mathbb{R}^d$, define:

$$\iota_C(x) = \begin{cases} 0 & x \in C \\ \infty & x \notin C \end{cases}$$

The Convex Indicator Function

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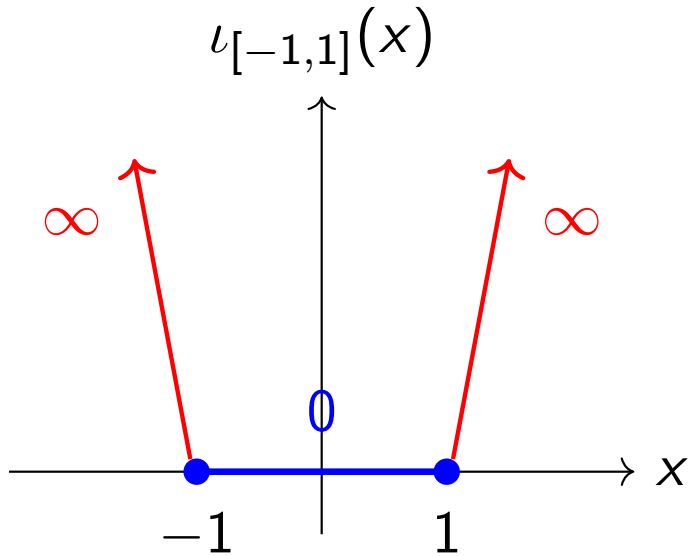
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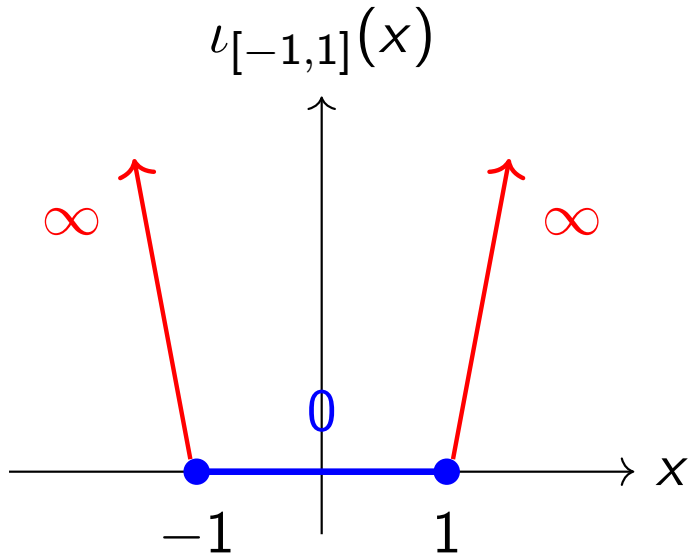


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Example: $\iota_{(0,1)}$ is NOT lsc — at $x = 0$, take $x_n = 1/n \to 0$ with $\iota_{(0,1)}(x_n) = 0$, but $\iota_{(0,1)}(0) = \infty$.

Constrained Optimization Using Indicator Functions

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1. **Unified framework:** Constrained and unconstrained have the same form
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3. **Algorithmic simplicity:** Proximal/subgradient methods work with this form

Example: Reformulation

Original:

$$\min_x \ \frac{1}{2}\|Ax - b\|_2^2 + \lambda\|x\|_1 \quad \text{s.t.} \quad x \geq 0$$

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To analyze optimality, we need to differentiate $\iota_{\mathbb{R}^d_+}$. This is where subdifferentials come in.

Review: First-Order Characterization of Convex Functions

For **differentiable** convex f:

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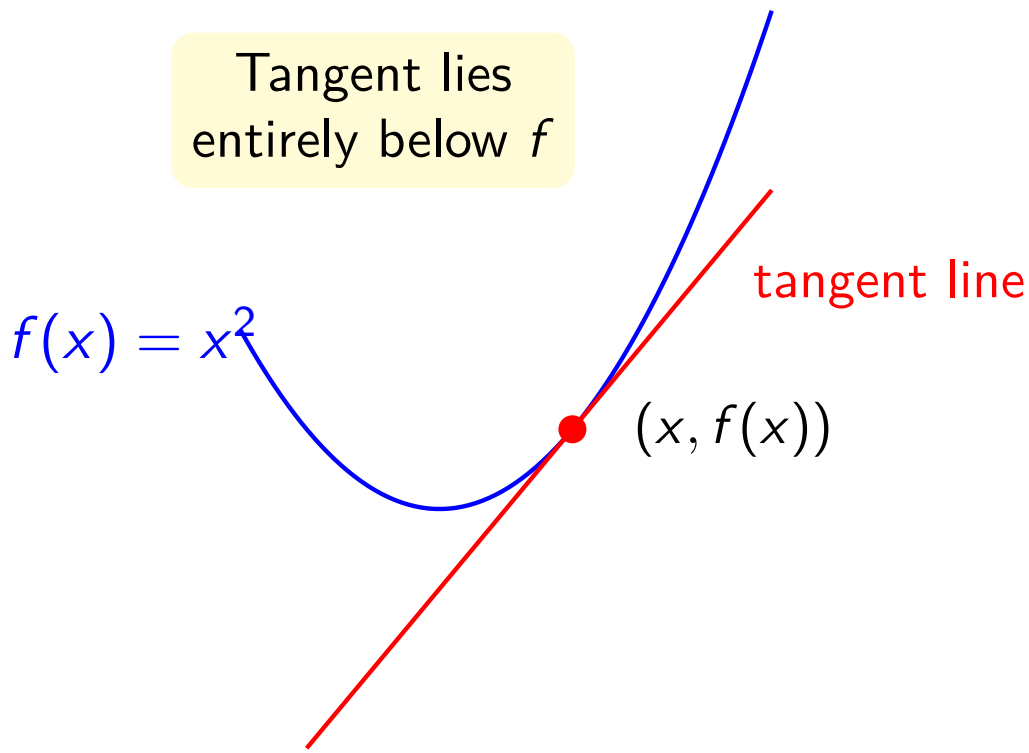
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For convex $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$, the **subdifferential** at $x \in \text{dom}(f)$ is:

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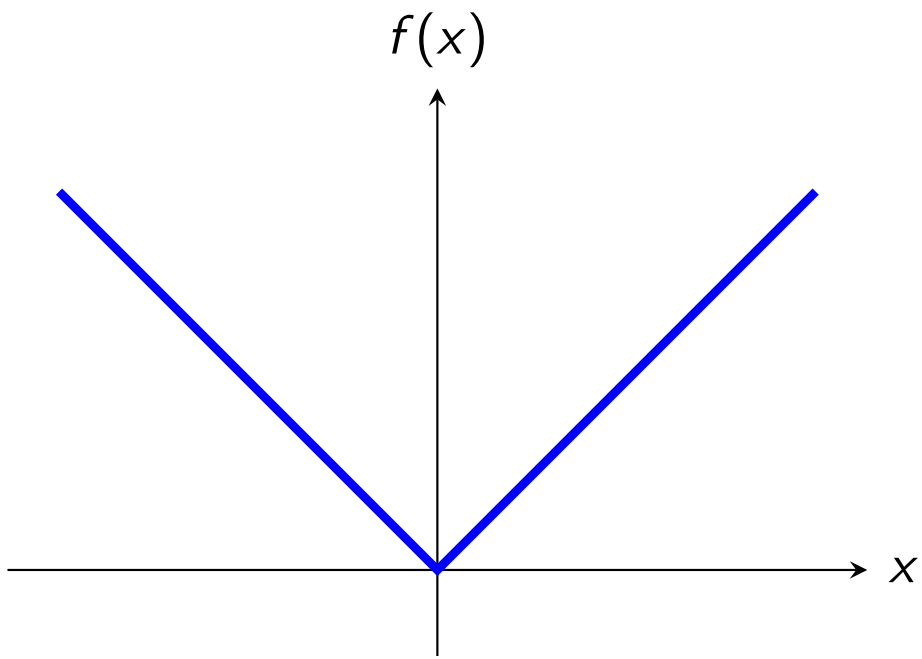
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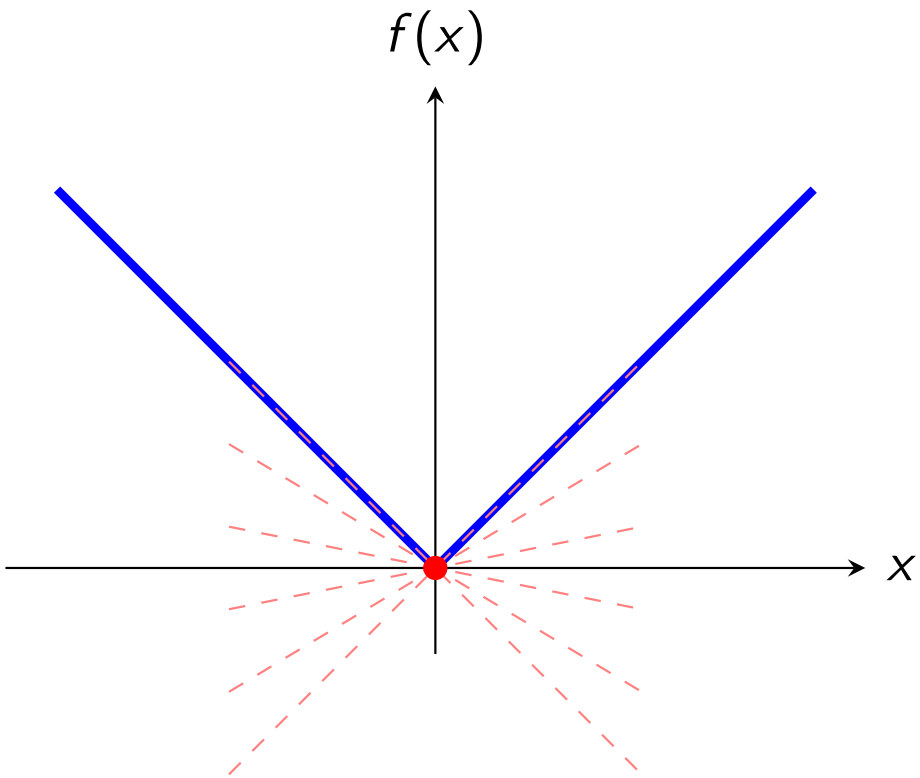
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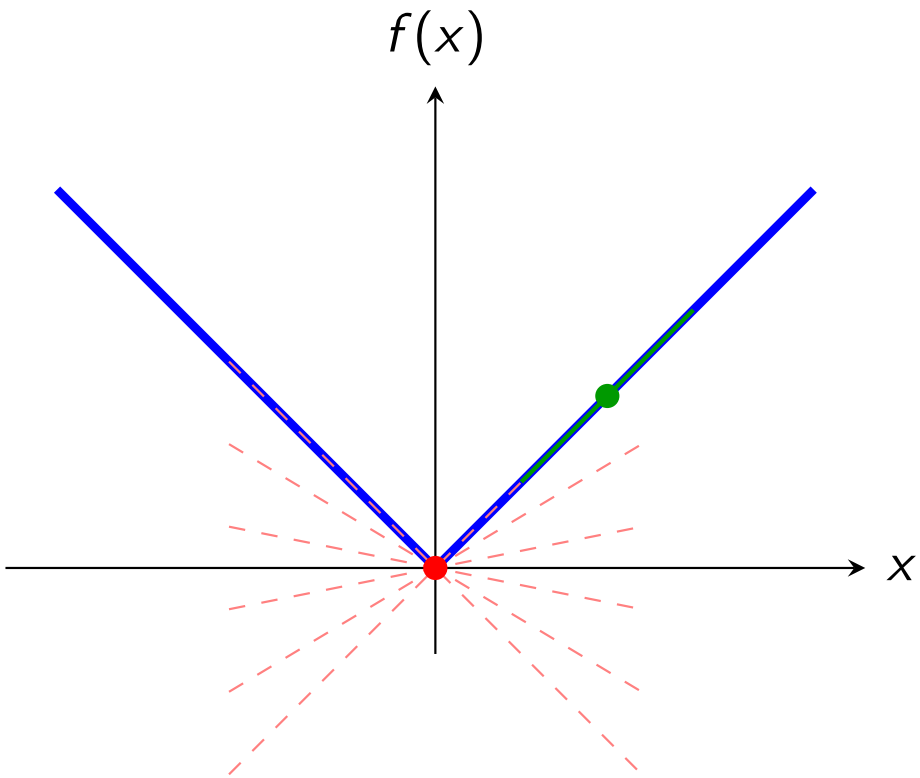
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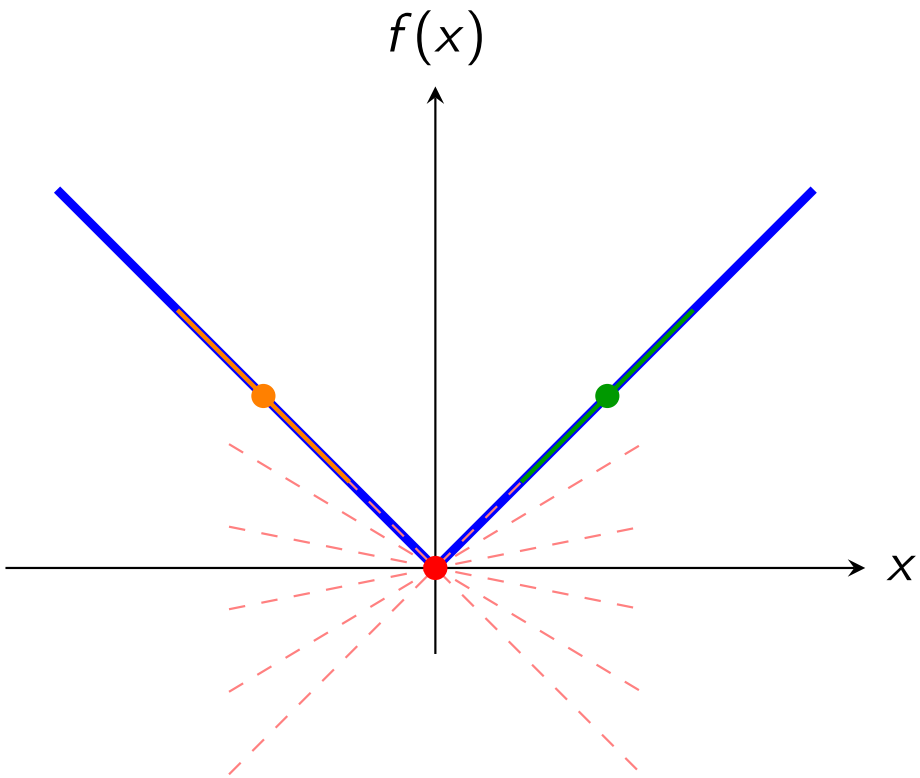
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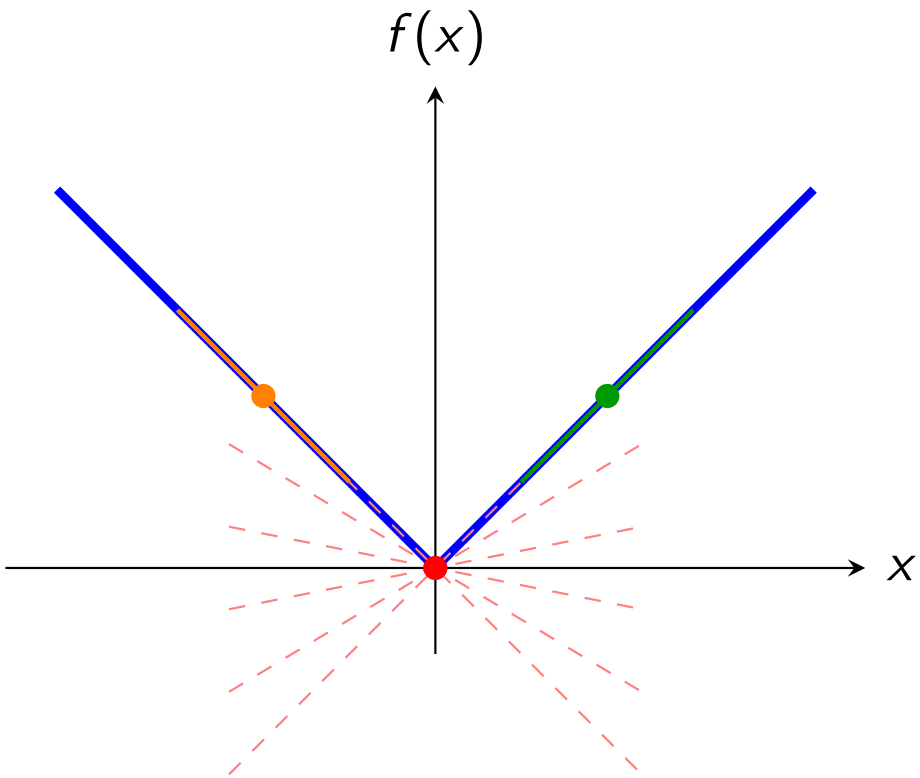


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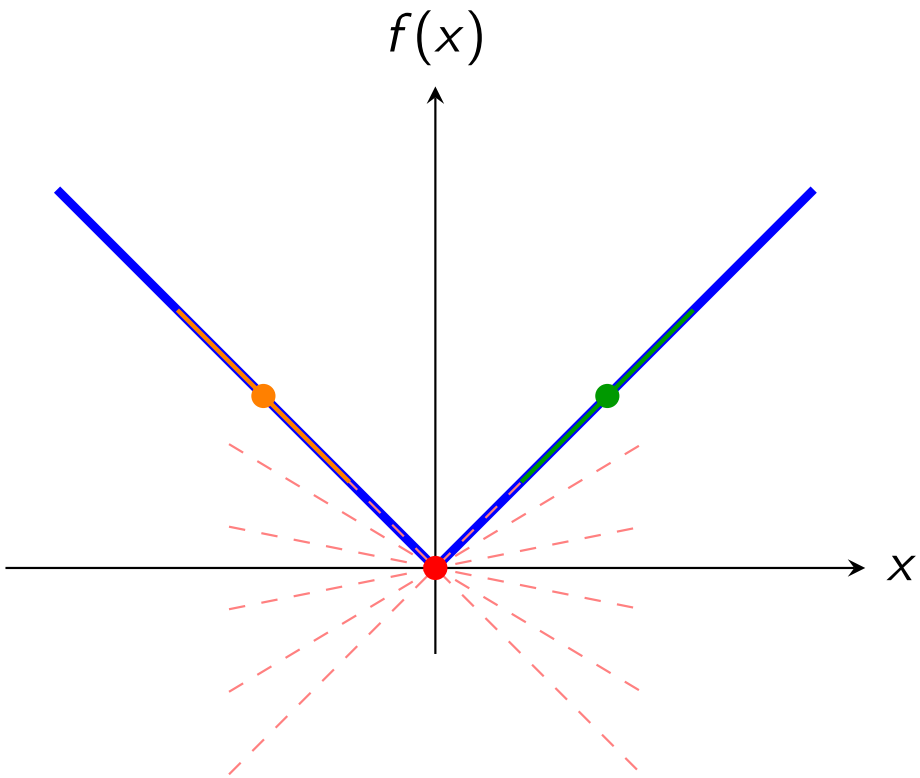
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- At kinks: subdifferential is an **interval** between left and right derivatives

This set-valued generalization is what lets us do calculus on nonsmooth functions.

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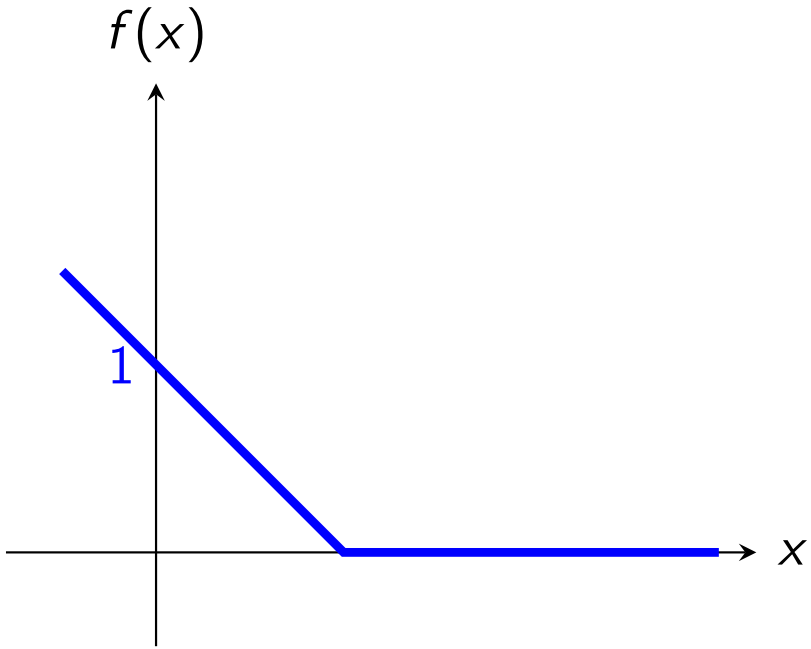
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Example: Subdifferential of $(1-x)_+$



Definition:

$$f(x) = \max\{0, 1-x\}$$

Subdifferential:

$$\partial f(x) = \begin{cases} \{-1\} & x < 1 \\ [-1, 0] & x = 1 \\ \{0\} & x > 1 \end{cases}$$

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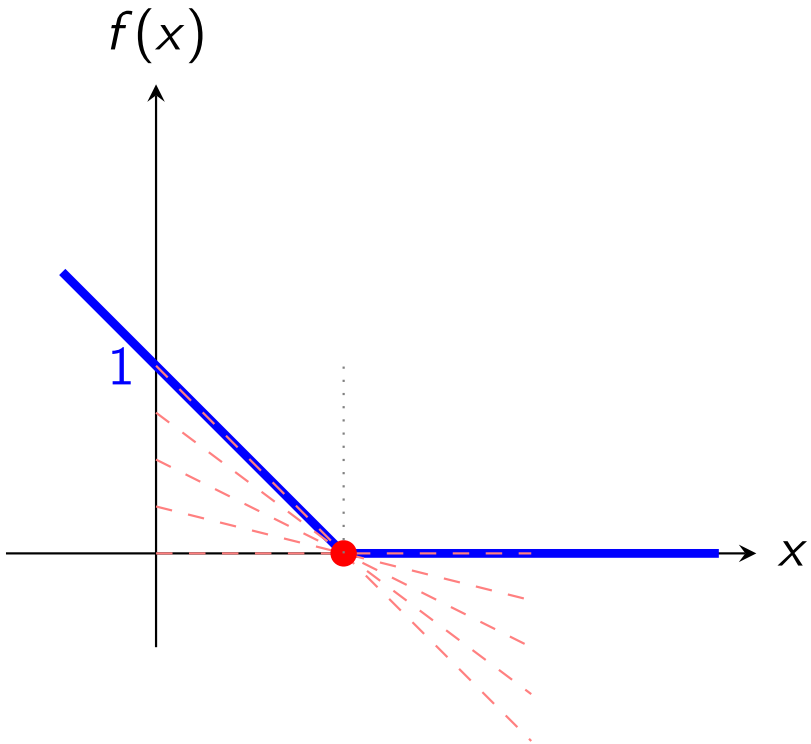
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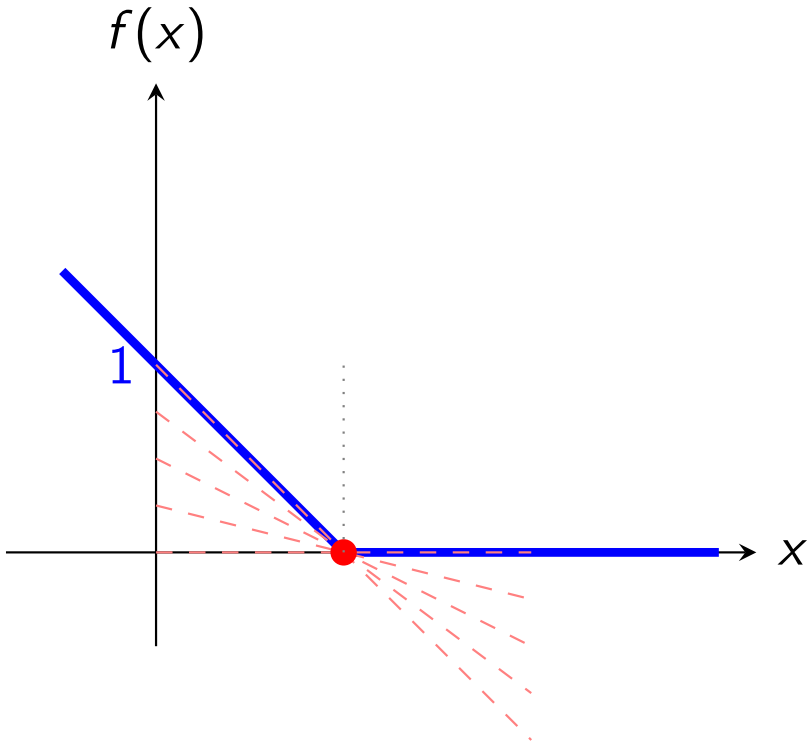
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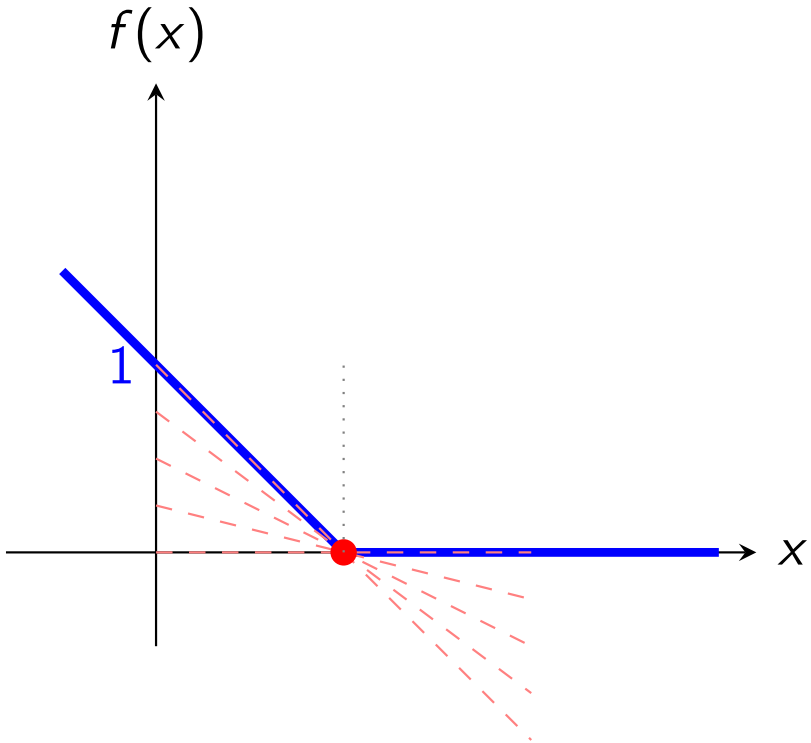
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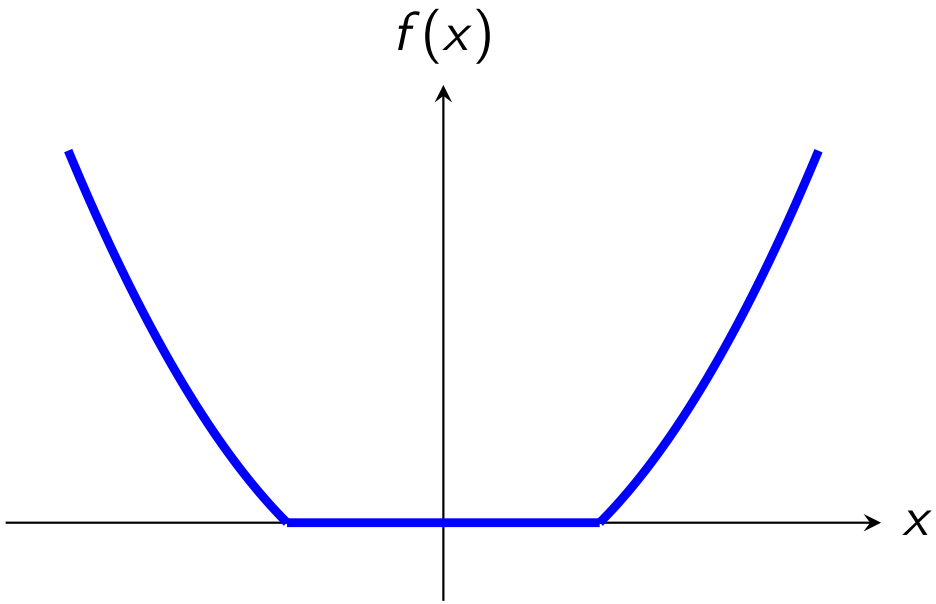
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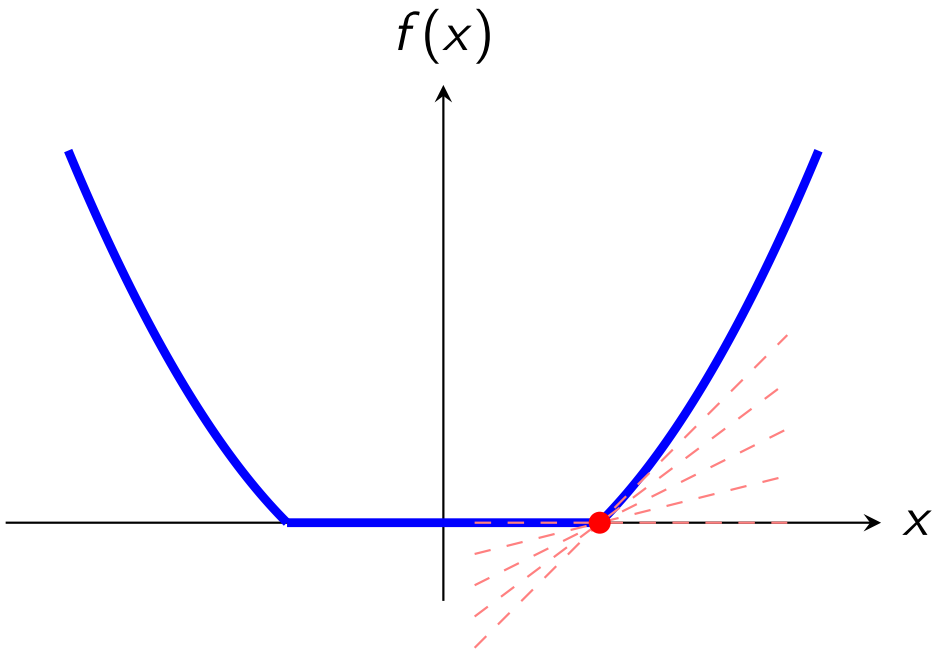
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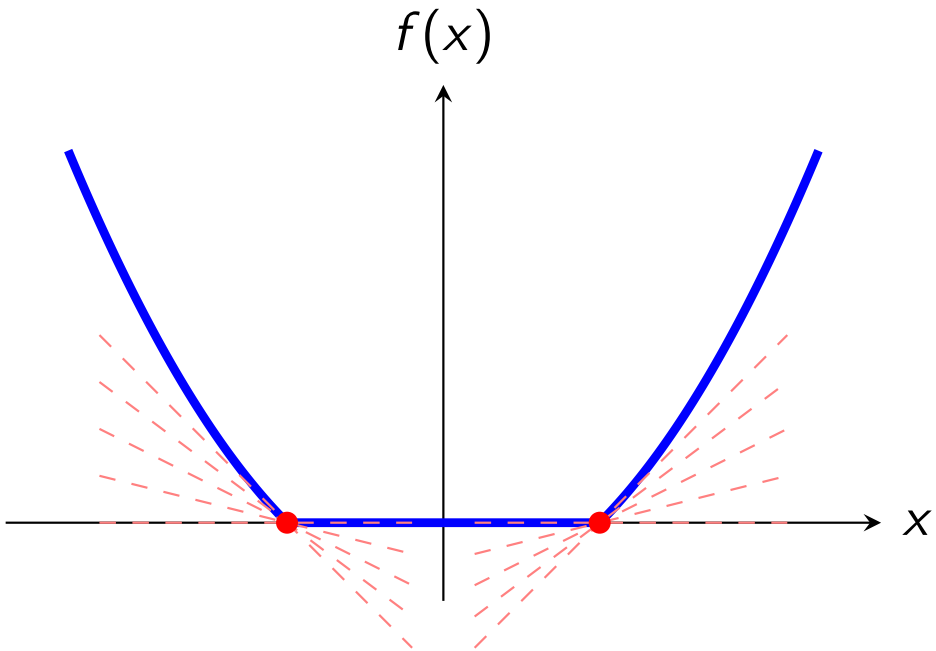
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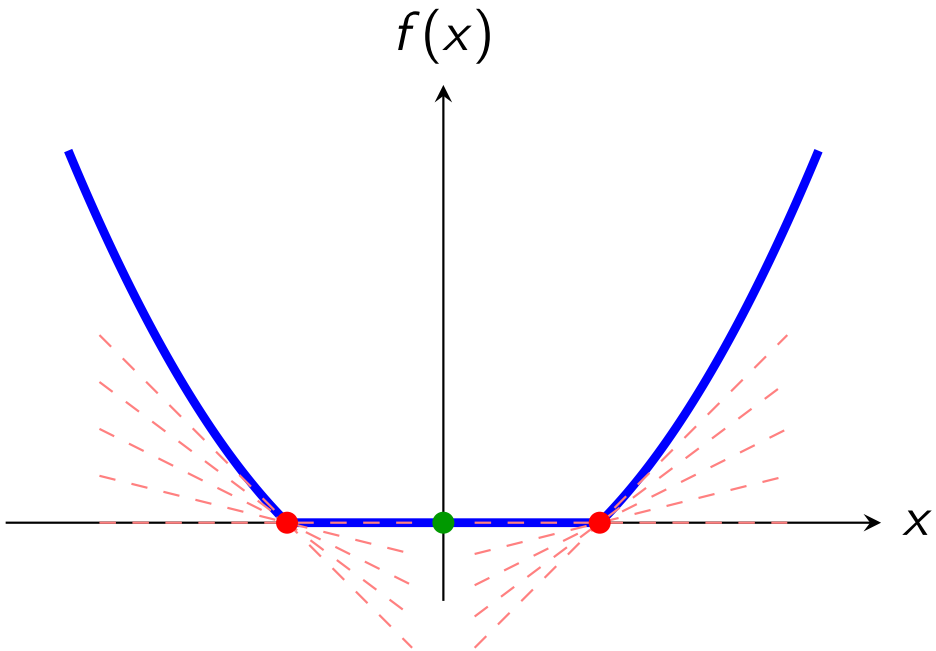
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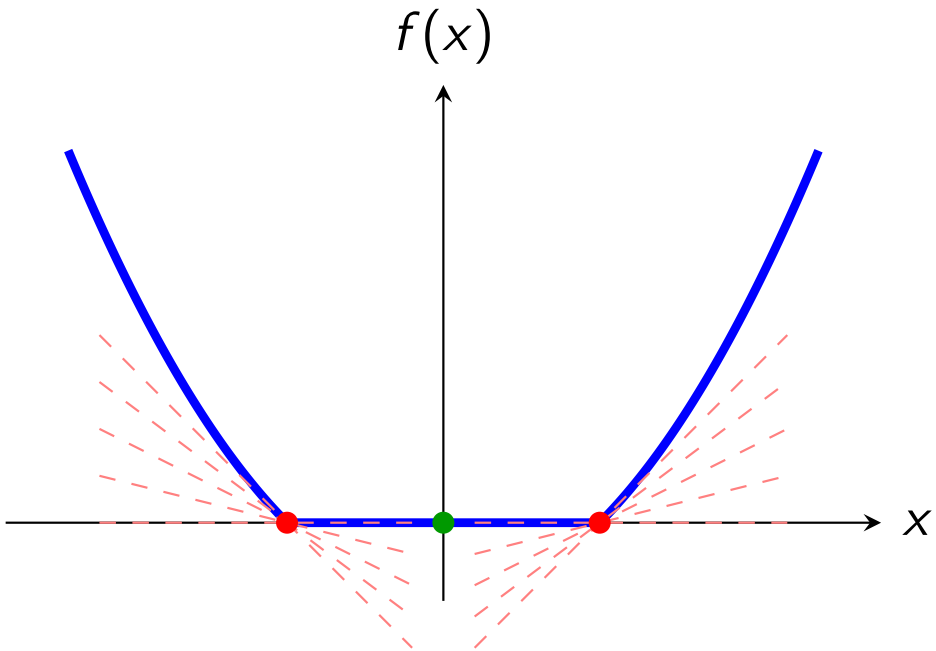
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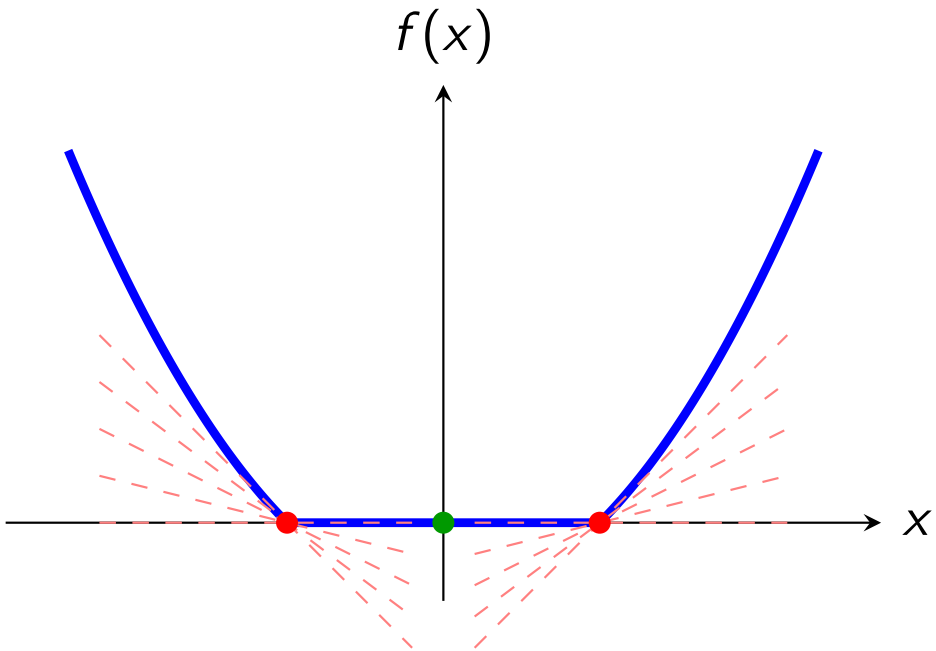


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Subdifferential of the ℓ_1 Norm, cont'd

Result

A vector g is a subgradient if its components satisfy:

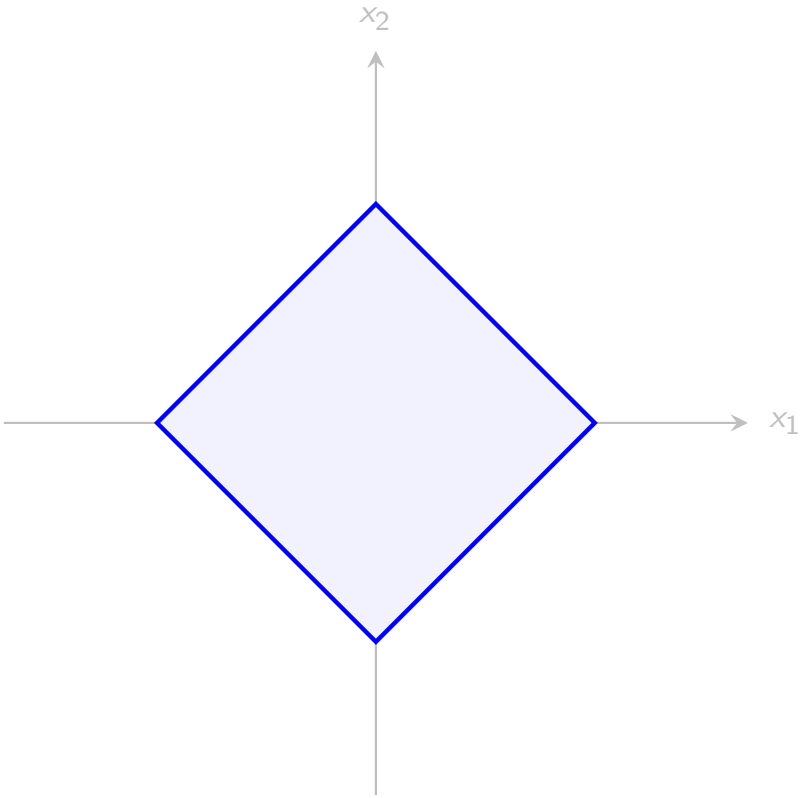
$$g_i \in \begin{cases} \{-1\} & x_i < 0 \\ [-1, 1] & x_i = 0 \\ \{1\} & x_i > 0 \end{cases}$$

Equivalently, the subdifferential is the **Cartesian product**:

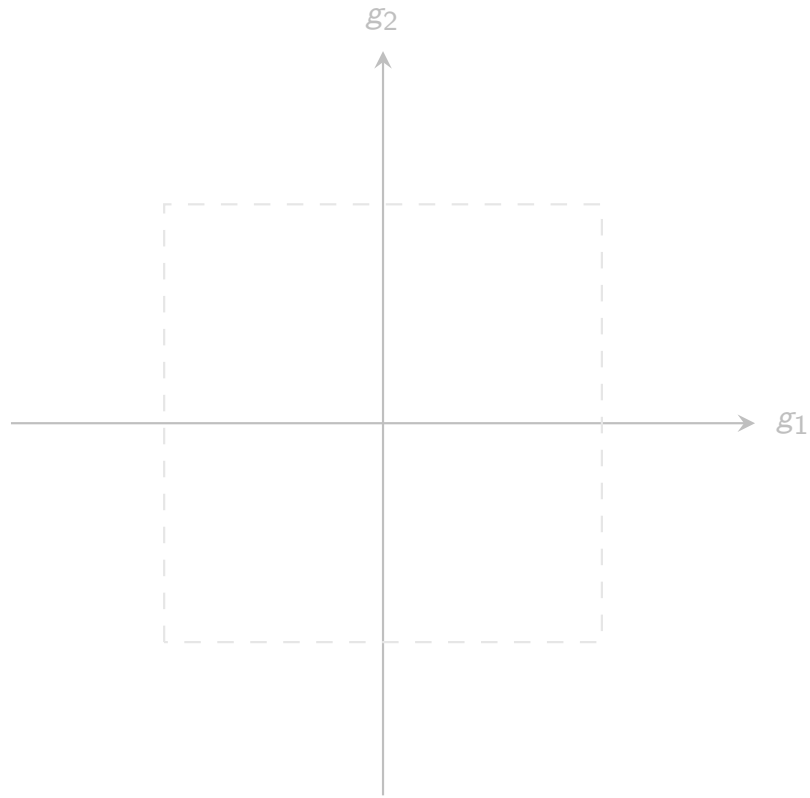
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ℓ_1 Subdifferential: Geometry in 2D

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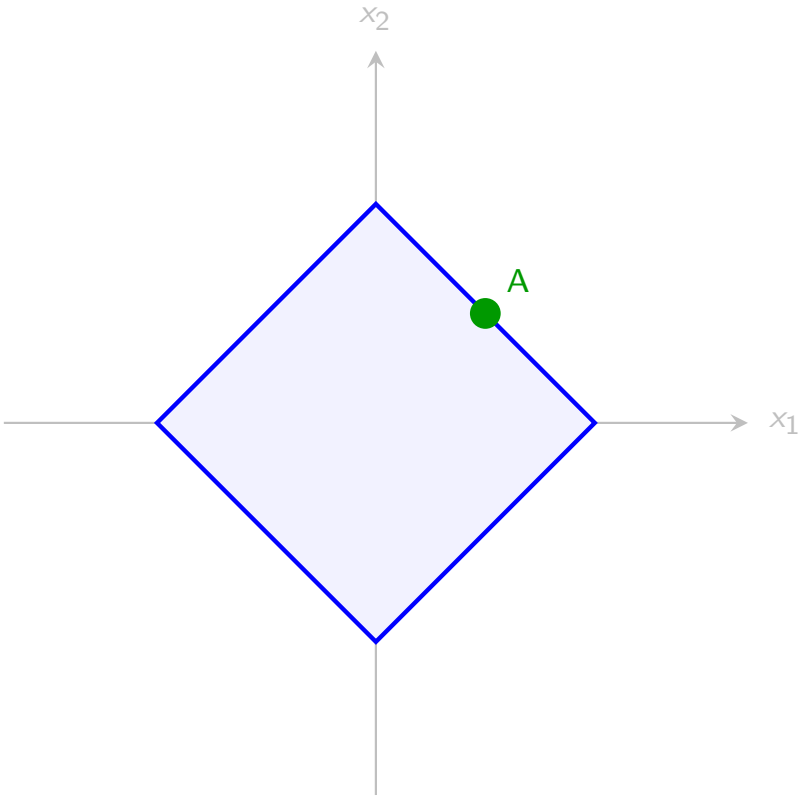


Dual Space ($g \in \partial \|x\|_1$)

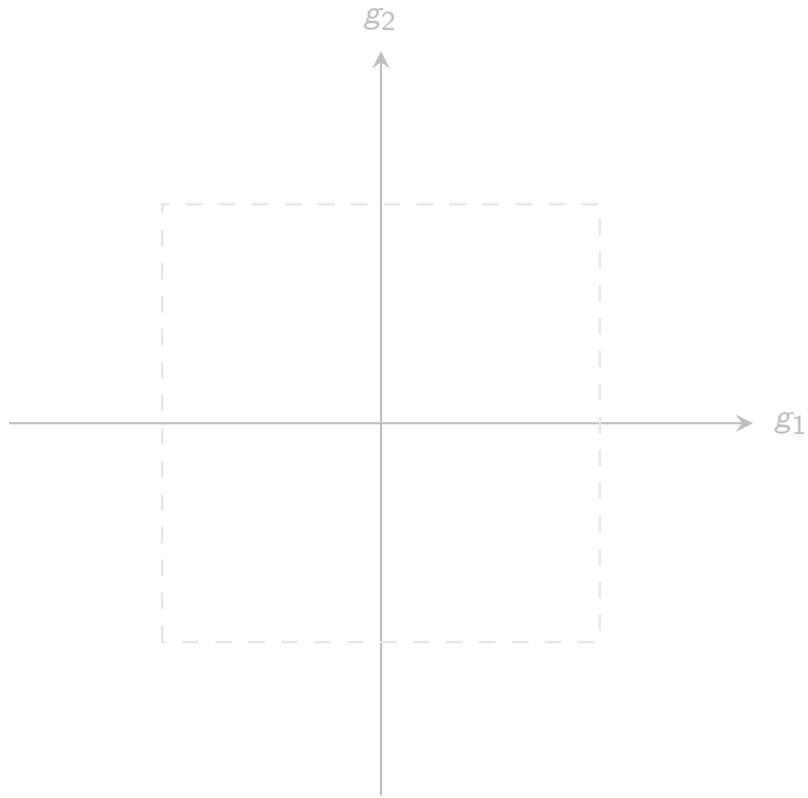


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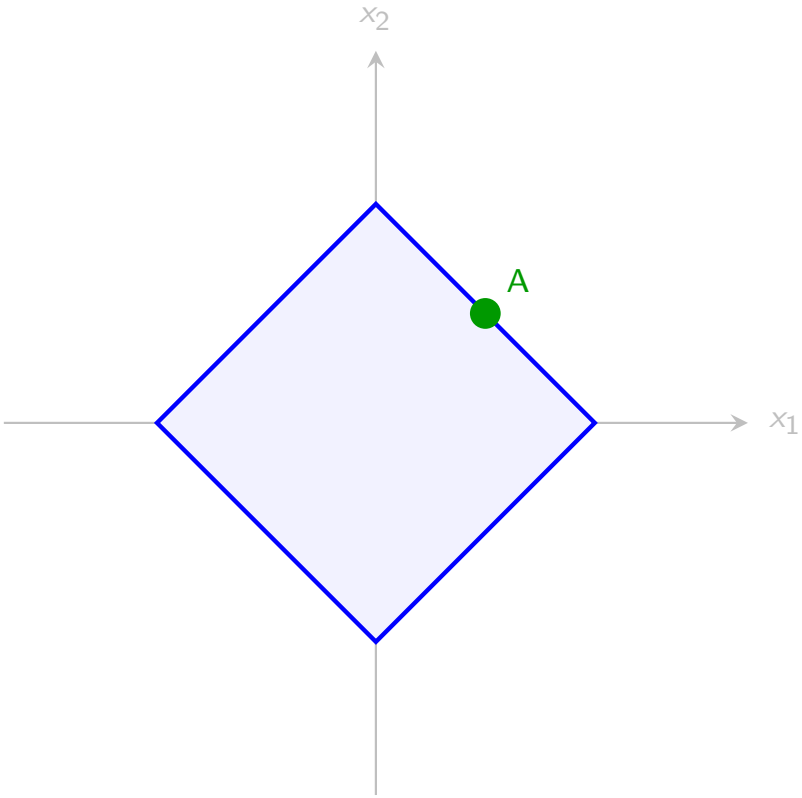
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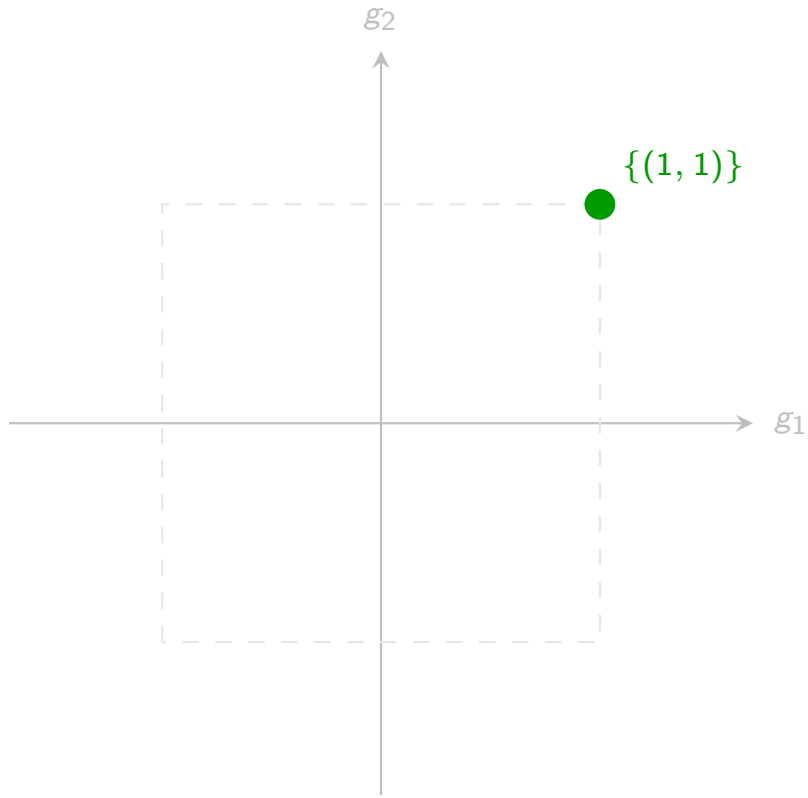
- **A**: Smooth edge ($x_1, x_2 > 0$)

ℓ_1 Subdifferential: Geometry in 2D

Primal Space (x)



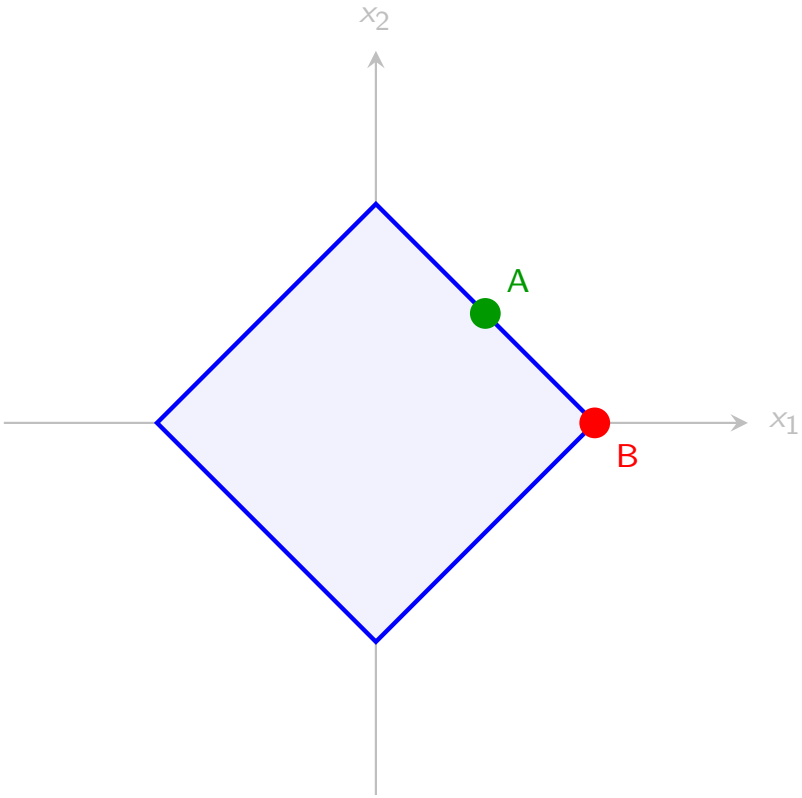
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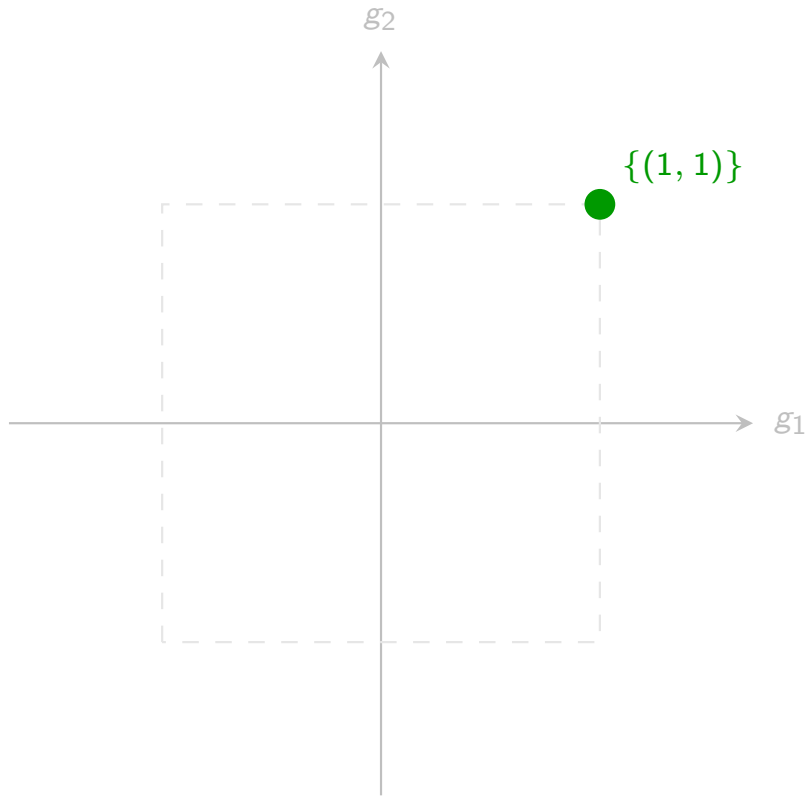
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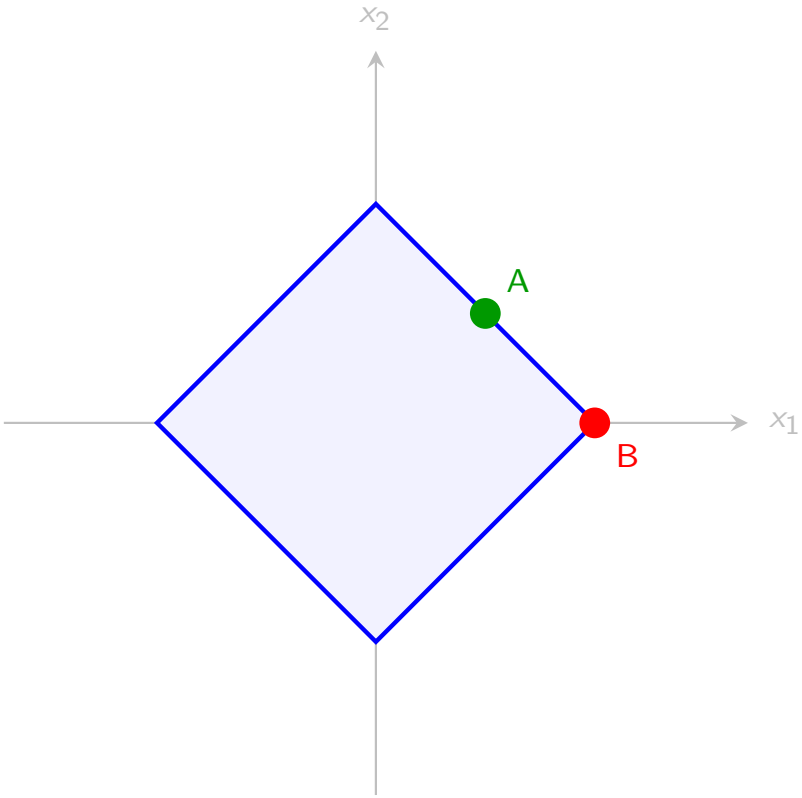
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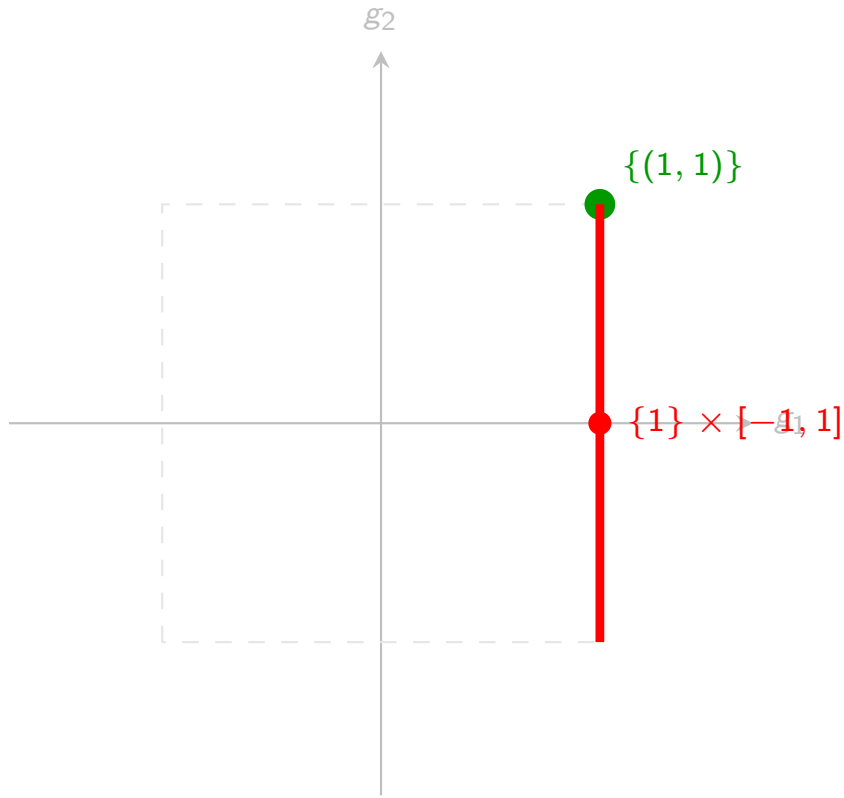
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ℓ_1 Subdifferential: Geometry in 2D

Primal Space (x)



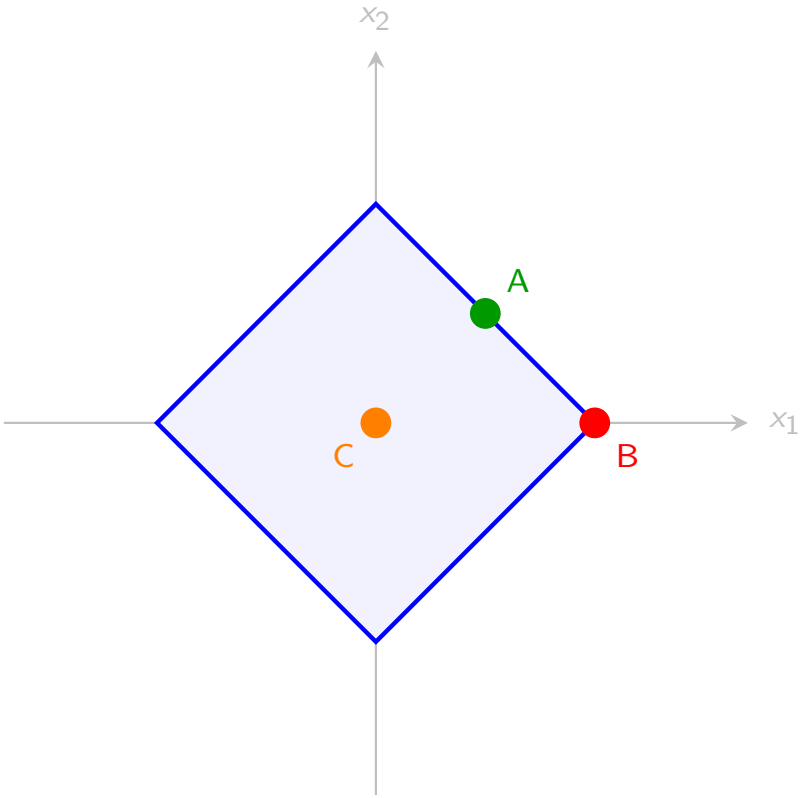
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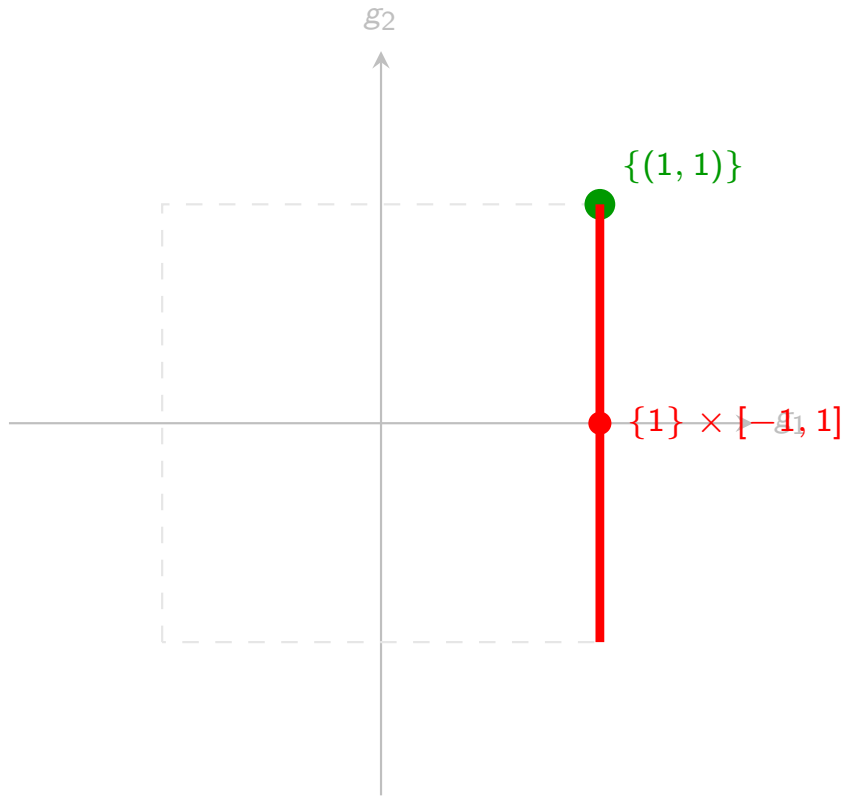
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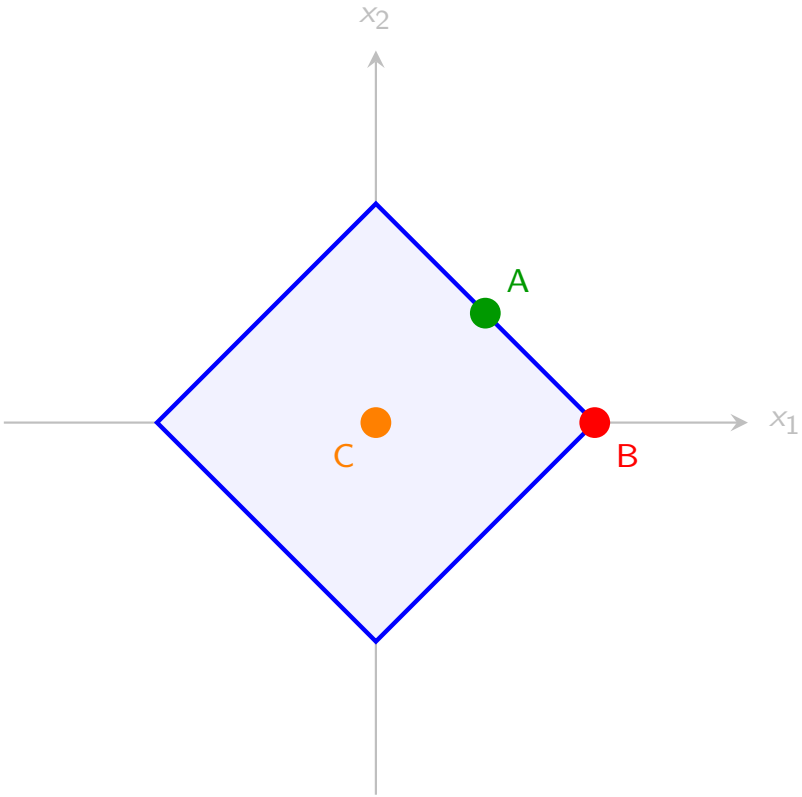
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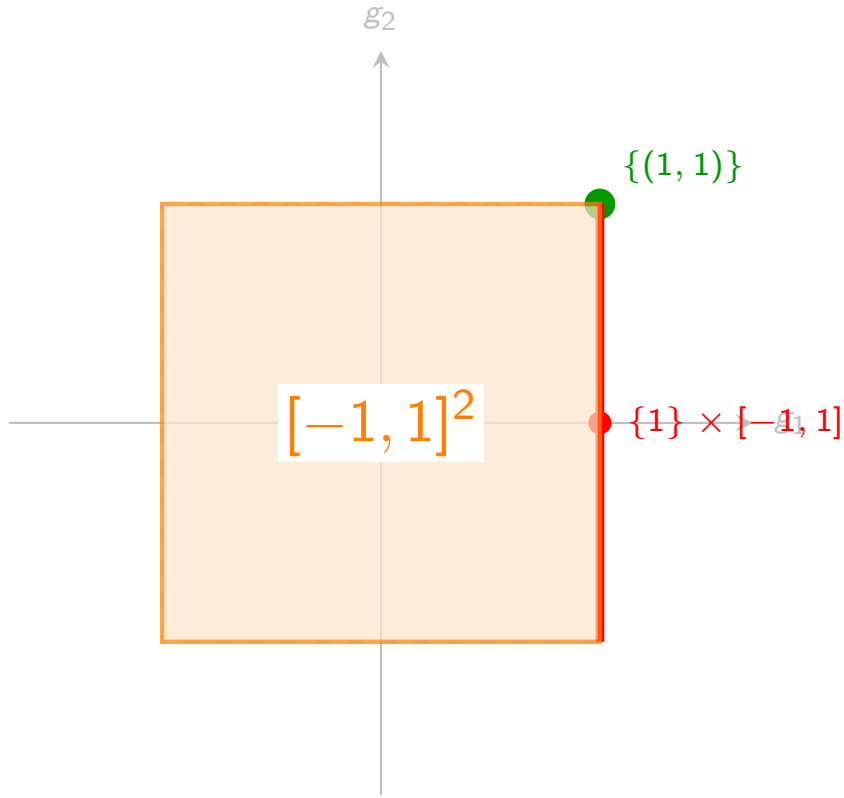
- **A**: Smooth edge $(x_1, x_2 > 0)$
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- **C**: Origin $(0, 0)$

ℓ_1 Subdifferential: Geometry in 2D

Primal Space (x)



Dual Space ($g \in \partial \|x\|_1$)



- **A**: Smooth edge ($x_1, x_2 > 0$)
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Geometric Insight

The subdifferential at the origin is the unit ball of the **dual norm** (ℓ_∞).

Example 4: Subdifferential of the ℓ_2 Norm

$$f(x) = \|x\|_2$$

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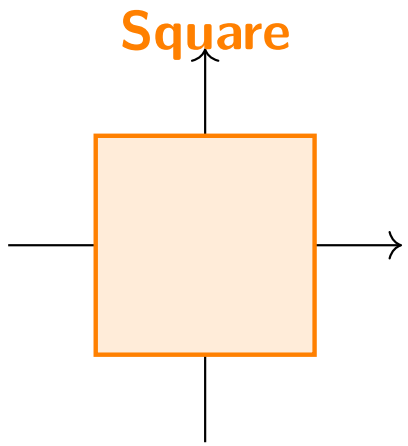
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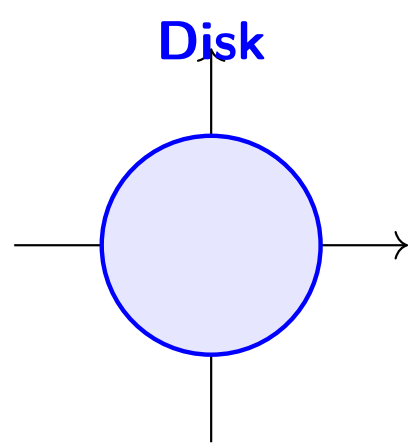
$$f(x) = \|x\|_2$$

- For $x \neq 0$: $\partial f(x) = \left\{ \dfrac{x}{\|x\|_2} \right\}$ (unit vector in direction of x)
- For $x = 0$: $\partial f(0) = \{v : \|v\|_2 \leq 1\}$ (closed unit ball)

Contrast with ℓ_1 at origin:



$\partial \|\cdot\|_1(0) = [-1, 1]^2$



$\partial \|\cdot\|_2(0) = B(0, 1)$

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What We Learned

1. **Lower semicontinuity** is essential for existence of minimizers — non-lsc functions may have infima that are not attained
2. **Extended-value functions** and **convex indicators** ι_C unify constrained and unconstrained optimization into a single framework
3. **Subdifferentials** $\partial f(x)$ generalize gradients to nonsmooth convex functions as the set of all supporting hyperplane slopes

Gradients vs Subdifferentials

Property	**Gradient** $\nabla f(x)$	**Subdifferential** $\partial f(x)$
Exists when	f differentiable	$x \in \text{int}(\text{dom}(f))$, f proper, lsc, convex
Size	single vector	closed convex set
Optimality	$\nabla f(x^*) = 0$	$0 \in \partial f(x^*)$
Algorithm	gradient descent	subgradient descent
Smoothness req.	yes	no
Geometric meaning	tangent hyperplane slope	set of supporting hyperplane slopes

Computational Note

While subdifferentials are **sets**, algorithms typically work with a **single element** from $\partial f(x)$ at each iteration.

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- At kinks: any convenient subgradient (e.g., a one-sided derivative)
- For ℓ_1 at zero coordinates: often $v_j = 0$ (minimum norm subgradient)

The choice of which element can affect algorithm behavior in practice, but **any choice is valid** for convergence guarantees.

Questions?

Next time: convex projections, subgradient calculus rules, optimality conditions