Machine Learning and Optimization

Lecture 6: Gradient Descent Convergence Theory

CSCI 4140/6960 — Spring 2026

Course website: `cs.rpi.edu/~gittea/.../mlandopt.html`

Learning Objectives

By the end of this lecture, you should be able to:

1. **Distinguish** three types of convergence and when each applies
2. **Define** strong convexity and prove its key properties
3. **Explain** why curvature enables parameter convergence
4. **Prove** convergence of GD for smooth convex functions at rate $O(1/T)$
5. **Prove** convergence of GD for strongly convex functions at rate $O((1 - 1/\kappa)^T)$
6. **Identify** when GD works well and when it doesn't (through examples)

Lecture Roadmap

Types of Convergence

Smoothness and Convergence

Strong Convexity

Convergence Results

Numerical Examples

Three Notions of Convergence

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What we'll prove today:

- Convex + smooth $\Rightarrow$ objective convergence $O(1/T)$
- **Strongly** convex + smooth $\Rightarrow$ parameter convergence $O((1 - 1/\kappa)^T)$

β-Smoothness

To prove GD converges, we need assumptions beyond convexity

Definition: f is **β-smooth** (or **β-Lipschitz continuous gradient**) if:

$$\boxed{\|\nabla f(\mathbf{w}) - \nabla f(\mathbf{v})\|_2 \leq \beta \|\mathbf{w} - \mathbf{v}\|_2 \quad \forall \mathbf{w}, \mathbf{v}}$$

Gradient is Lipschitz continuous with constant β

Equivalent condition (for twice differentiable f):

$$\nabla^2 f(\mathbf{w}) \preceq \beta I \quad \forall \mathbf{w}$$

All eigenvalues of Hessian $\leq \beta$

Intuition:

- Gradient doesn't change too rapidly
- Function curvature bounded
- No arbitrarily steep regions

Example: $f(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T Q \mathbf{w}$ is β-smooth with $\beta = \lambda_{\max}(Q)$

Geometric Intuition: β-Smooth vs Non-Smooth

What does β-smoothness look like geometrically?

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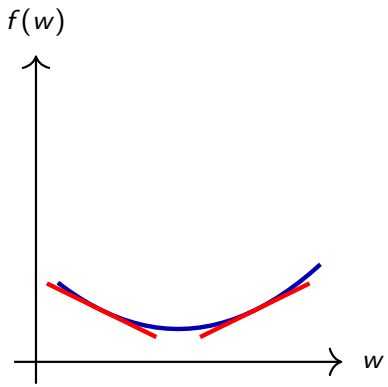
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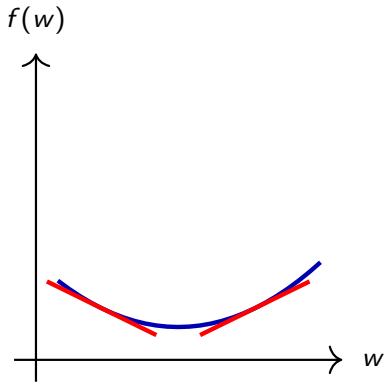
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β-smooth

Gradients change gradually

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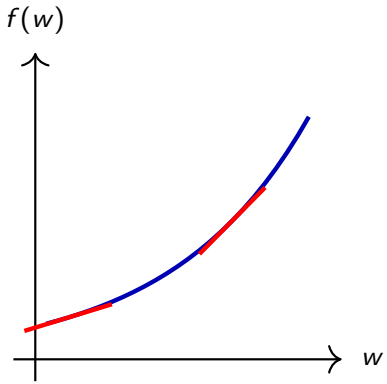
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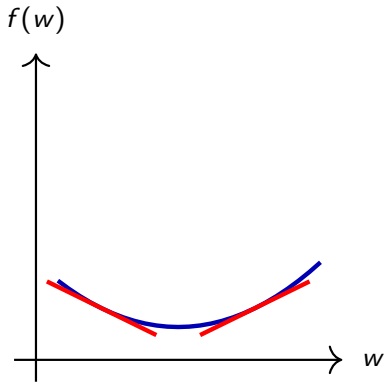
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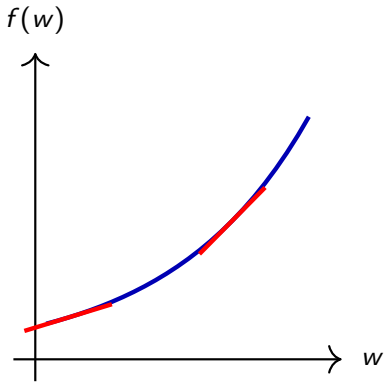
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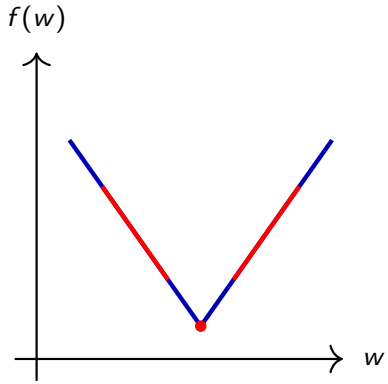
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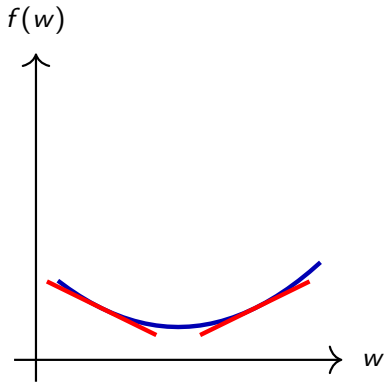
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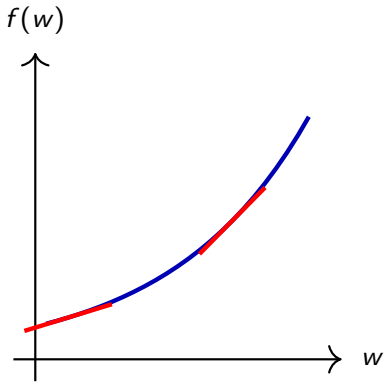
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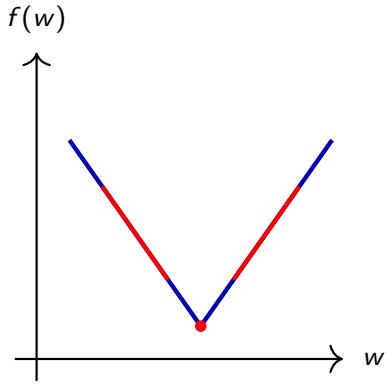
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Key Insight

β-smoothness requires: (1) **Differentiable** everywhere AND (2) **Bounded curvature** ($\nabla^2 f \preceq \beta I$). Exponential e^w fails (2), absolute value $|w|$ fails (1).

Why Smoothness Matters: Quadratic Upper Bound

Lemma: If f is β-smooth, then:

$$\boxed{f(\mathbf{v}) \leq f(\mathbf{w}) + \langle \nabla f(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle + \frac{\beta}{2} \|\mathbf{v} - \mathbf{w}\|_2^2}$$

First-order Taylor plus quadratic correction!

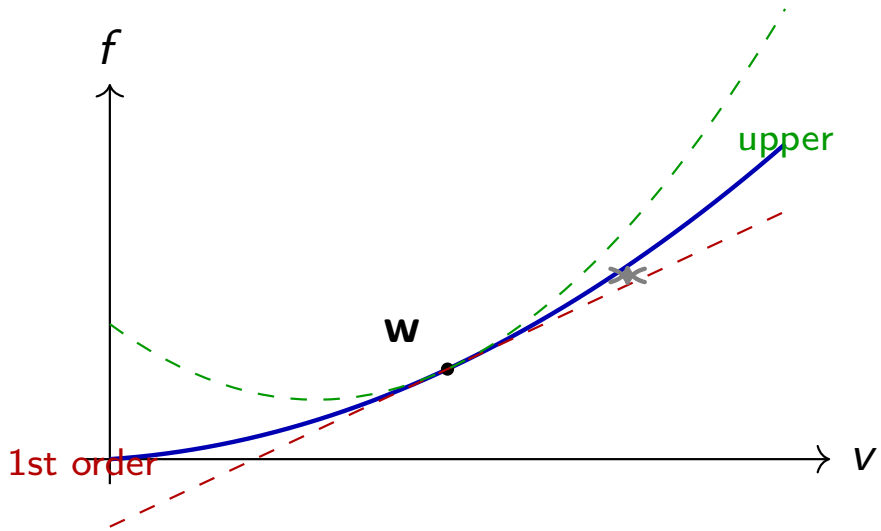
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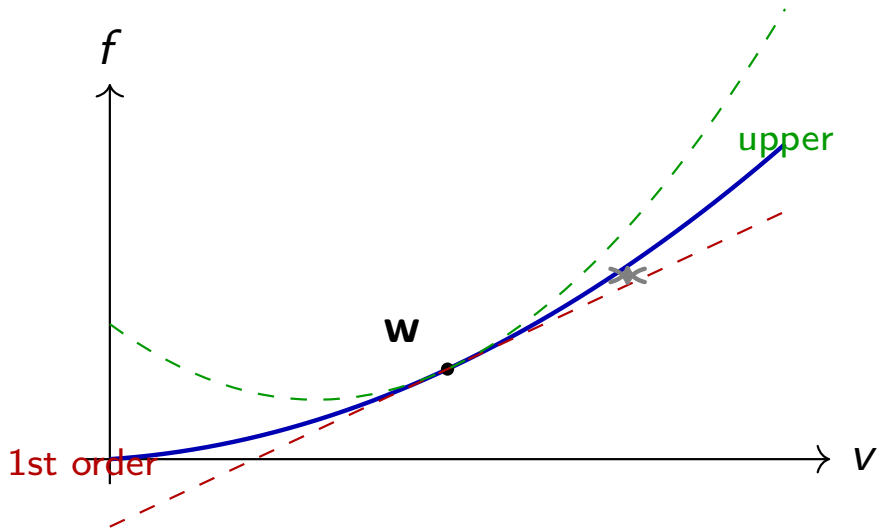
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Key insight:

- First-order Taylor (linear) **underestimates** convex f
- True function $f(v)$ sandwiched between
- Quadratic upper bound controls approximation error
- Error grows at most $O(\beta \|v - w\|^2)$

This is why we can trust gradient descent steps!

Quadratic Upper Bound Proof

Starting from:

$$f(\mathbf{v}) - f(\mathbf{w}) = \int_0^1 \langle \nabla f(\mathbf{w} + t(\mathbf{v} - \mathbf{w})), \mathbf{v} - \mathbf{w} \rangle \, dt$$

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By Cauchy-Schwarz and β-smoothness:

$$|\langle \nabla f(\mathbf{w} + t(\mathbf{v} - \mathbf{w})) - \nabla f(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle| \leq \beta t \|\mathbf{v} - \mathbf{w}\|_2^2$$

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Therefore:

$$f(\mathbf{v}) - f(\mathbf{w}) \leq \langle \nabla f(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle + \int_0^1 \beta t \|\mathbf{v} - \mathbf{w}\|_2^2 \, dt = \langle \nabla f(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle + \frac{\beta}{2} \|\mathbf{v} - \mathbf{w}\|_2^2$$

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Proof: Apply the quadratic upper bound lemma with:

$$\mathbf{v} = \mathbf{w}_t = \mathbf{w}_{t-1} - \frac{1}{\beta}\nabla f(\mathbf{w}_{t-1})$$

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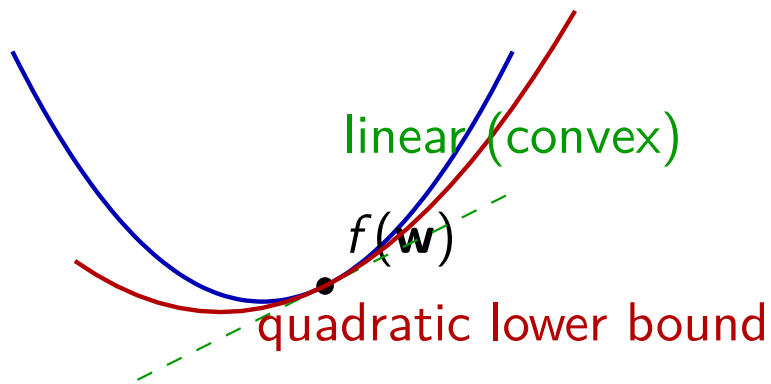
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Example

$f(\mathbf{w}) = \frac{1}{2}\mathbf{w}^T Q\mathbf{w}$ with $Q \succ 0$ is μ-strongly convex with $\mu = \lambda_{\min}(Q)$
Since $\nabla^2 f(\mathbf{w}) = Q$, and $Q \succeq \lambda_{\min}(Q)I$

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When f is both μ-strongly convex AND β-smooth:

- Smoothness: $\nabla^2 f(\mathbf{w}) \preceq \beta I$ (curvature bounded above)
- Strong convexity: $\nabla^2 f(\mathbf{w}) \succeq \mu I$ (curvature bounded below)

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Measures problem difficulty:

- $\kappa = 1$: Perfect sphere (isotropic), easiest
- $\kappa \gg 1$: Elongated ellipsoid, harder

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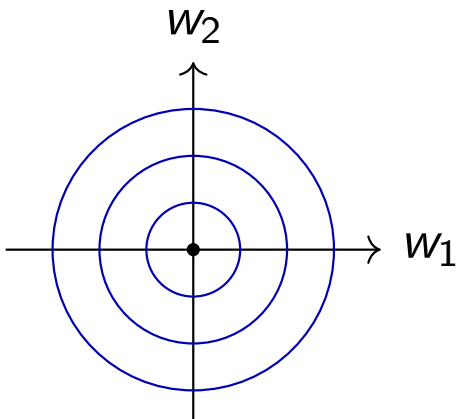
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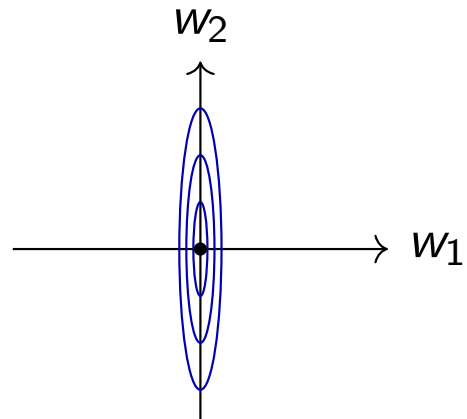
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Convergence Theorem (Convex Case)

Theorem: Suppose f is convex and β-smooth. Run GD with constant step size $\alpha = 1/\beta$:

$$\mathbf{w}_t = \mathbf{w}_{t-1} - \frac{1}{\beta}\nabla f(\mathbf{w}_{t-1})$$

Then after T iterations:

$$\boxed{f(\mathbf{w}_T) - f(\mathbf{w}^*) \leq \frac{\beta\|\mathbf{w}_0 - \mathbf{w}^*\|_2^2}{2T}}$$

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Interpretation:

- **Convergence type**: Objective gap only, rate $O(1/T)$ (sublinear)
- **Iteration complexity**: $T = O(\beta/\varepsilon)$ iterations for ε-suboptimality
- Faster for smaller β (less curved)
- Faster if $\mathbf{w}_0$ close to $\mathbf{w}^*$

Convergence Theorem (Convex Case)

Theorem: Suppose f is convex and β-smooth. Run GD with constant step size $\alpha = 1/\beta$:

$$\mathbf{w}_t = \mathbf{w}_{t-1} - \frac{1}{\beta}\nabla f(\mathbf{w}_{t-1})$$

Then after T iterations:

$$\boxed{f(\mathbf{w}_T) - f(\mathbf{w}^*) \leq \frac{\beta\|\mathbf{w}_0 - \mathbf{w}^*\|_2^2}{2T}}$$

Interpretation:

- **Convergence type**: Objective gap only, rate $O(1/T)$ (sublinear)
- **Iteration complexity**: $T = O(\beta/\varepsilon)$ iterations for ε-suboptimality
- Faster for smaller β (less curved)
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Important

No parameter convergence guarantee! Without strong convexity, $\|\mathbf{w}_T - \mathbf{w}^*\|$ may not converge to zero.

Convergence for Strongly Convex Functions - Statement

[1] This rate is for using $\alpha = \frac{1}{\beta}$. You can do slightly better: see Appendix.

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1	0	0	1 step converges!
2	0.5	$\approx 10^{-30}$	Very fast
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For $\kappa = 100$: need $T \approx 460$ iterations for 99% error reduction

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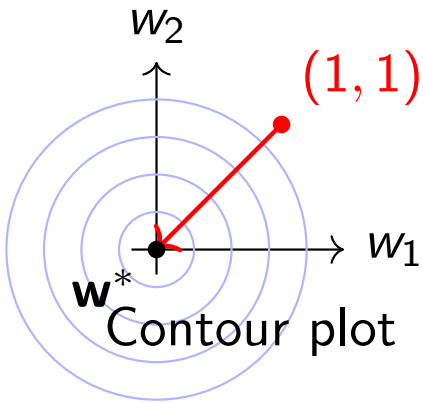
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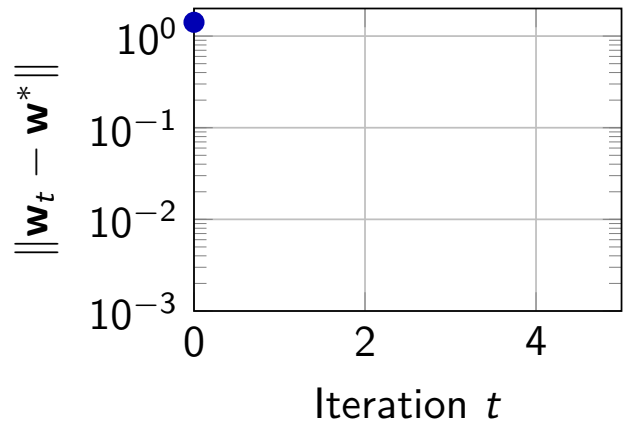
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Contour plot

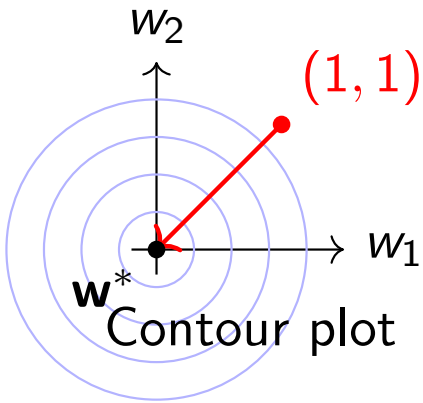


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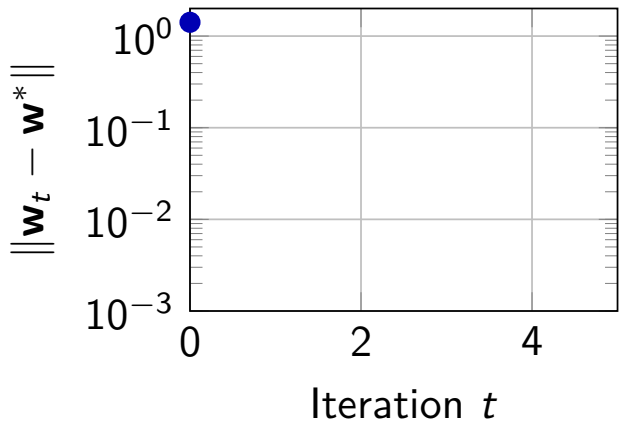
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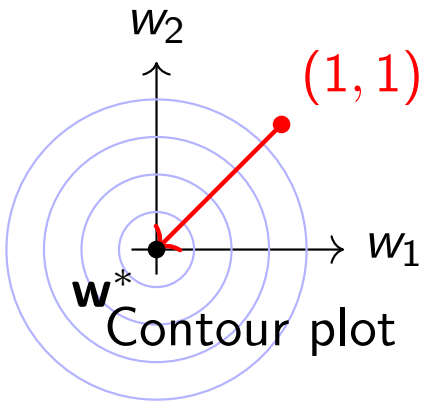
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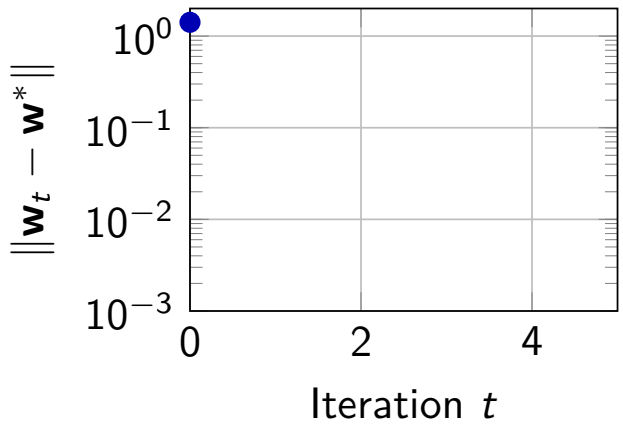
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Observed: Exact convergence in 1 iteration ✓

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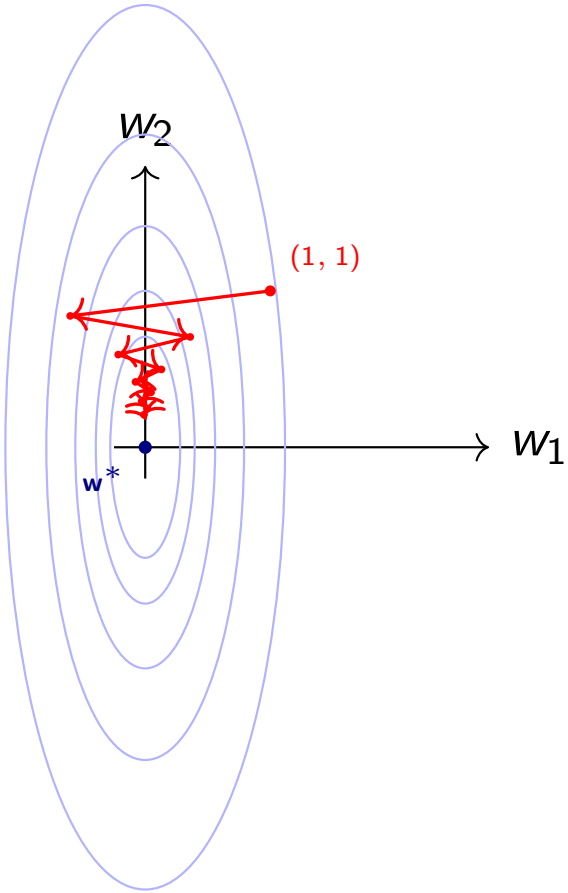
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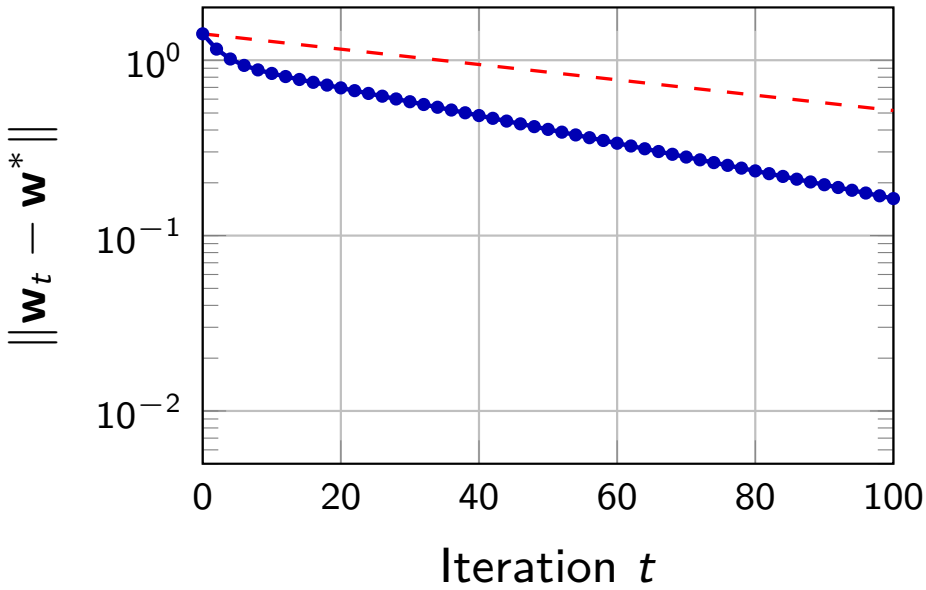
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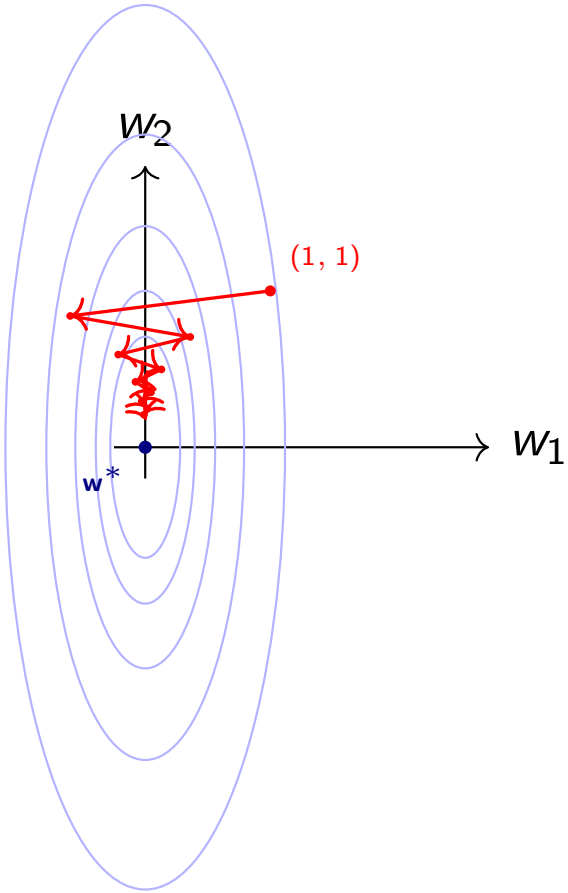


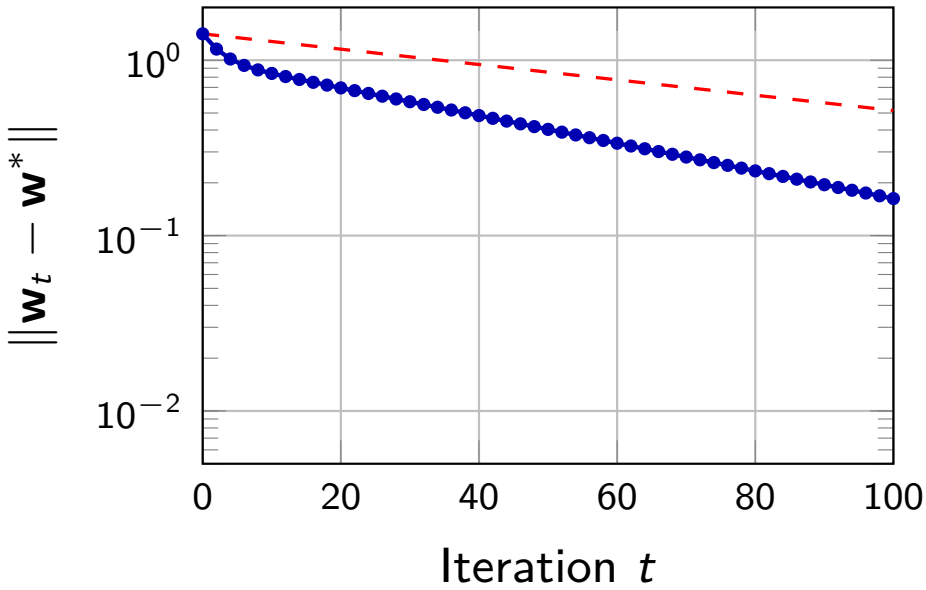
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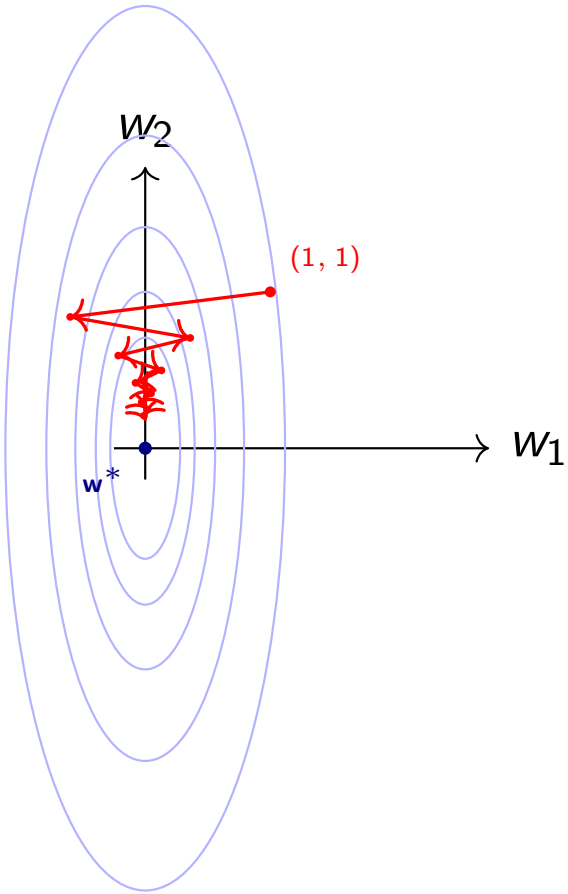
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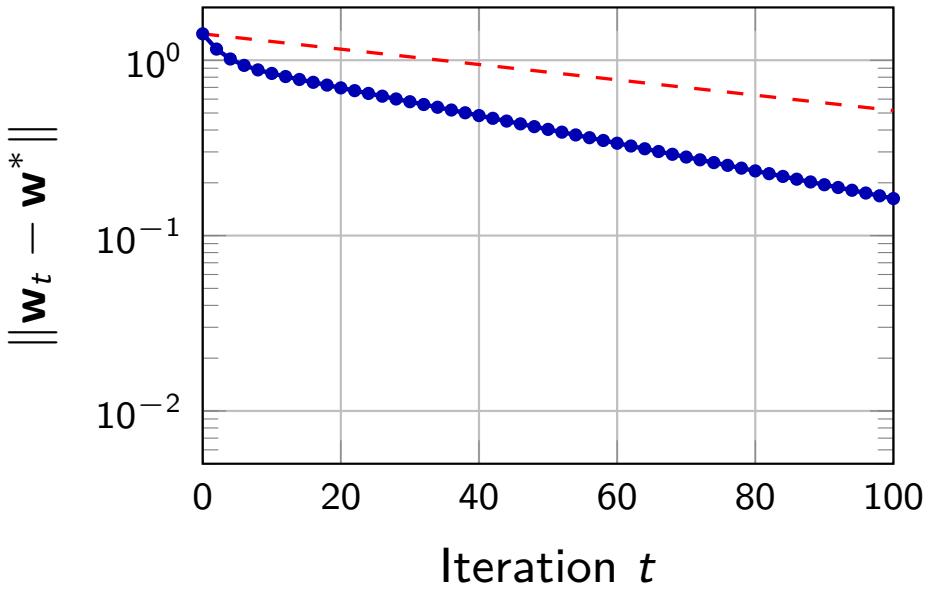
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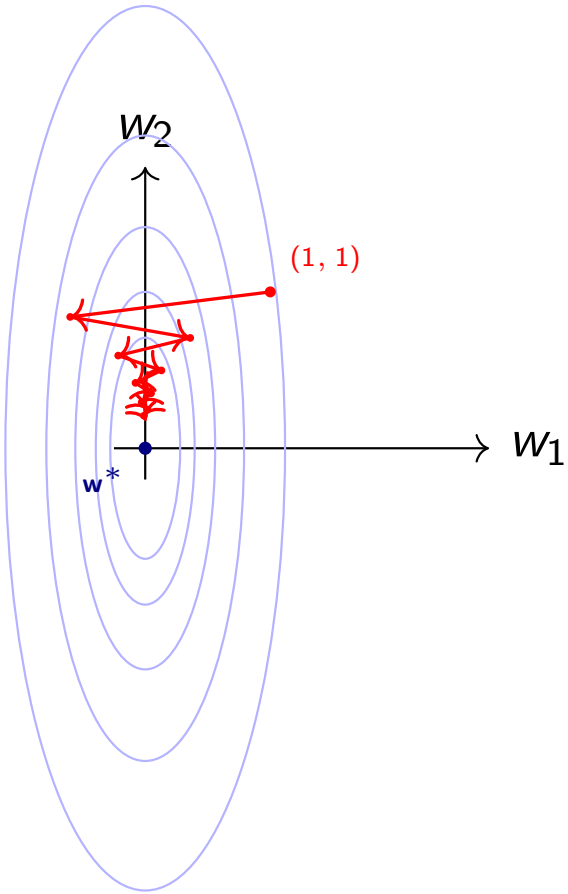
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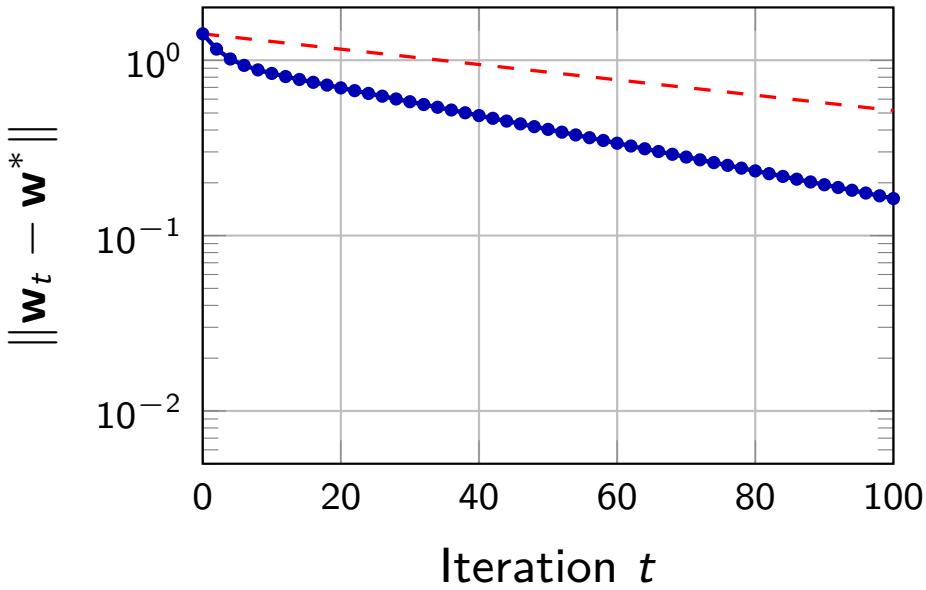
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Next lecture: Stochastic Gradient Descent uses random mini-batches!

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Coming Next

- **Lecture 7:** Stochastic Gradient Descent (random sampling for large n)
- **Lecture 8:** Subgradient methods (when f not differentiable)

Questions?

Proof of Convergence Theorem (Convex Case) - Proof Idea

We want to upper bound the final objective gap

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after T gradient steps.

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When we add them, the drops in squared distance *telescope*— meaning they cancel out—, giving

$$\begin{aligned}\sum_{t=1}^{T} f(\mathbf{w}_t) - f(\mathbf{w}^*) &\lesssim \|\mathbf{w}_0 - \mathbf{w}^*\|^2 - \|\mathbf{w}_T - \mathbf{w}^*\|^2 \\ &\leq \|\mathbf{w}_0 - \mathbf{w}^*\|^2\end{aligned}$$

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Finally, monotonic decrease of $f(\mathbf{w}_t)$ lets us turn the *sum* of gaps into a bound on the *last iterate*.

Proof of Convergence Theorem (Convex Case) - Part 1

By convexity at $\mathbf{w}_{t-1}$:

$$f(\mathbf{w}_{t-1}) - f(\mathbf{w}^*) \leq \langle \nabla f(\mathbf{w}_{t-1}), \mathbf{w}_{t-1} - \mathbf{w}^* \rangle$$

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Rearranging the inner product:

$$\langle \nabla f(\mathbf{w}_{t-1}), \mathbf{w}_{t-1} - \mathbf{w}^* \rangle = \frac{1}{2\eta} \left(\|\mathbf{w}_{t-1} - \mathbf{w}^*\|_2^2 - \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 \right) + \frac{\eta}{2} \|\nabla f(\mathbf{w}_{t-1})\|_2^2$$

Proof of Convergence Theorem (Convex Case) - Part 2

Plug into the convexity bound:

$$f(\mathbf{w}_{t-1}) - f(\mathbf{w}^*) \leq \frac{1}{2\eta}\left(\|\mathbf{w}_{t-1} - \mathbf{w}^*\|_2^2 - \|\mathbf{w}_t - \mathbf{w}^*\|_2^2\right) + \frac{\eta}{2}\|\nabla f(\mathbf{w}_{t-1})\|_2^2$$

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This is the inequality we will sum over t.

Proof of Convergence Theorem (Convex Case) - Part 3

Sum the boxed inequality from $t = 1$ to T:

$$\sum_{t=1}^{T} \left(f(\mathbf{w}_t) - f(\mathbf{w}^*) \right) \leq \frac{1}{2\eta} \sum_{t=1}^{T} \left(\|\mathbf{w}_{t-1} - \mathbf{w}^*\|_2^2 - \|\mathbf{w}_t - \mathbf{w}^*\|_2^2 \right)$$

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Therefore each term in the sum is at least $f(\mathbf{w}_T) - f(\mathbf{w}^*)$, giving

$$T\big(f(\mathbf{w}_T) - f(\mathbf{w}^*)\big) \leq \sum_{t=1}^{T} \big(f(\mathbf{w}_t) - f(\mathbf{w}^*)\big).$$

Proof of Convergence Theorem (Convex Case) - Part 4

Combine the last observation with the last boxed inequality:

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Recall that $\eta = 1/\beta$:

$$f(\mathbf{w}_T) - f(\mathbf{w}^*) \leq \frac{\beta\|\mathbf{w}_0 - \mathbf{w}^*\|_2^2}{2T}$$

□

Rates for GD on strongly convex smooth functions

Assume f is μ-strongly convex and β-smooth (so $\kappa = \frac{\beta}{\mu}$) with minimizer $\mathbf{w}^*$ and optimum value f^*.

Theorem 1: Exponential objective value convergence.
GD with step size $\alpha = \frac{1}{\beta}$ satisfies:

$$\boxed{f(\mathbf{w}_T) - f^* \leq \left(1 - \frac{1}{\kappa}\right)^T \left(f(\mathbf{w}_0) - f^*\right)}$$

Theorem 2: Exponential parameter convergence.
GD with the *optimal* constant step size

$$\alpha^\star = \frac{2}{\mu + \beta}$$

satisfies:

$$\boxed{\|\mathbf{w}_T - \mathbf{w}^*\|_2 \leq \left(\frac{\kappa - 1}{\kappa + 1}\right)^T \|\mathbf{w}_0 - \mathbf{w}^*\|_2}$$

Comparing the Results

1. Since $\mu < \beta$, we have $\mu + \beta < 2\beta$, hence:

$$\alpha^\star = \frac{2}{\mu + \beta} > \frac{1}{\beta} = \alpha$$

   Knowing both μ and β enables larger, more aggressive step sizes when trying to recover parameters.

2. *Parameters can be derived at a (slightly) faster rate.*

$$\frac{\kappa - 1}{\kappa + 1} \leq \frac{\kappa - 1}{\kappa} = 1 - \frac{1}{\kappa}.$$

Obvious question: Can the objective convergence rate be (slightly) improved to match the parameter convergence rate? No! Both convergence rates are optimal.

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PL Inequality

If f is μ-strongly convex with minimizer $\mathbf{w}^*$, then:

$$\boxed{\frac{1}{2}\|\nabla f(\mathbf{w})\|_2^2 \geq \mu(f(\mathbf{w}) - f(\mathbf{w}^*))}$$

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Interpretation:

- Gradient norm measures distance to optimum (in function values)
- If gradient is small, we must be close to optimal function value
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Just as quadratic upper bound (smoothness) enables monotonic descent, PL inequality (strong convexity) enables **exponential/linear convergence**!

Proof of the PL inequality

Strong convexity says every tangent line sits below the function, with a quadratic gap. For all $\mathbf{v}$, we have

$$\begin{aligned} f(\mathbf{v}) &\geq f(\mathbf{w}) + \langle \nabla f(\mathbf{w}), \mathbf{v} - \mathbf{w} \rangle + \frac{\mu}{2} \|\mathbf{v} - \mathbf{w}\|_2^2 \\ &\geq \min_{\mathbf{z}} \left(f(\mathbf{w}) + \langle \nabla f(\mathbf{w}), \mathbf{z} - \mathbf{w} \rangle + \frac{\mu}{2} \|\mathbf{z} - \mathbf{w}\|_2^2 \right). \end{aligned}$$

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Minimizing the right-hand side (check this!) gives

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At $\mathbf{v} = \mathbf{v}^*$, we get that

$$f^* \geq f(\mathbf{w}) - \frac{1}{2\mu} \|\nabla f(\mathbf{w})\|_2^2.$$

Rearranging gives the PL inequality. □

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Iterating T times gives the result. $\square$

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Goal: Prove parameter convergence directly using linear algebra

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Setup:

- $f : \mathbb{R}^d \to \mathbb{R}$ is μ-strongly convex and β-smooth
- Unique minimizer $\mathbf{w}^*$ with $\nabla f(\mathbf{w}^*) = 0$
- GD with constant step size α: $\mathbf{w}_{t+1} = \mathbf{w}_t - \alpha \nabla f(\mathbf{w}_t)$
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Proof sketch: Express the gradient in terms of an "averaged Hessian" along the line segment from $\mathbf{w}^*$ to $\mathbf{w}_t$. Show this averaged Hessian contracts the error at each iteration.

The Averaged Hessian

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By fundamental theorem of calculus applied to ∇f:

$$\nabla f(\mathbf{w}) - \nabla f(\mathbf{w}^*) = \int_0^1 \nabla^2 f(\mathbf{w}^* + t(\mathbf{w} - \mathbf{w}^*))(\mathbf{w} - \mathbf{w}^*)\, dt$$

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Observe that since $\mu I \preceq \nabla^2 f(\mathbf{w}) \preceq \beta I$ for all $\mathbf{w}$,

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I.e., the integral of PSD matrices with eigenvalues in $[\mu, \beta]$ has eigenvalues in $[\mu, \beta]$.

GD as Matrix Iteration

Let $H_t := H(\mathbf{w}_t)$ so that

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Because H_t is symmetric with eigenvalues in $[\mu, \beta]$,

$$\|\mathbf{e}_{t+1}\|_2 \leq \|I - \alpha H_t\|_2 \, \|\mathbf{e}_t\|_2 = \left(\max_{\lambda \in [\mu, \beta]} |1 - \alpha \lambda| \right) \|\mathbf{e}_t\|_2.$$

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$$q(\alpha) := \max_{\lambda \in [\mu, \beta]} |1 - \alpha\lambda|$$

Then:

$$\|\mathbf{e}_{t+1}\|_2 \leq q(\alpha)\|\mathbf{e}_t\|_2$$

Iterating T times:

$$\boxed{\|\mathbf{w}_T - \mathbf{w}^*\|_2 \leq q(\alpha)^T \|\mathbf{w}_0 - \mathbf{w}^*\|_2}$$

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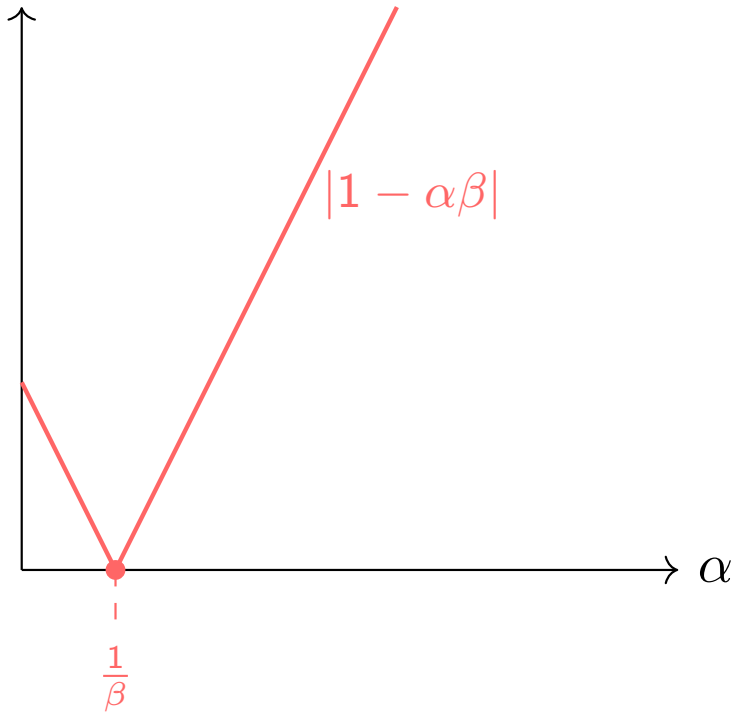
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Since $|1 - \alpha\lambda|$ is convex in λ, the maximum on $[\mu, \beta]$ occurs at an endpoint:

$$q(\alpha) = \max\{|1 - \alpha\mu|, |1 - \alpha\beta|\}$$

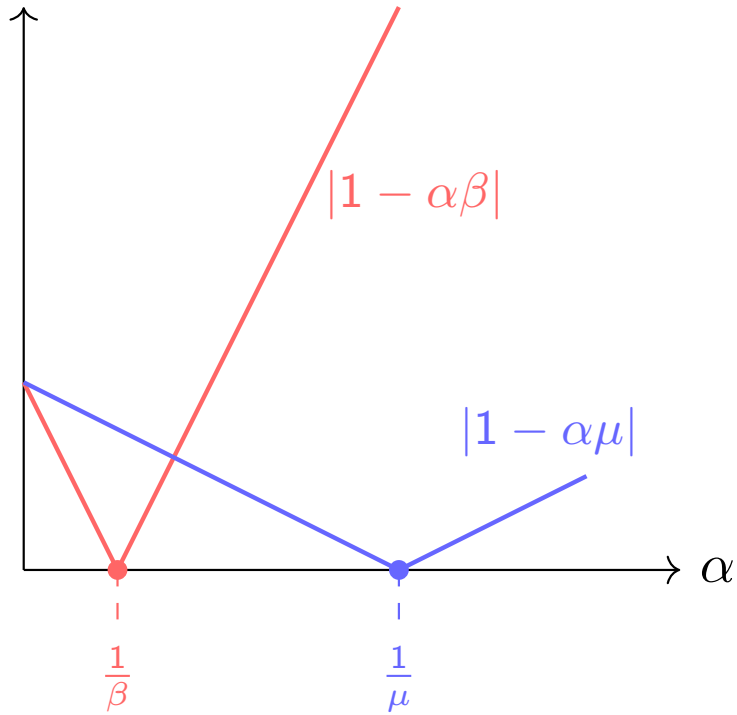
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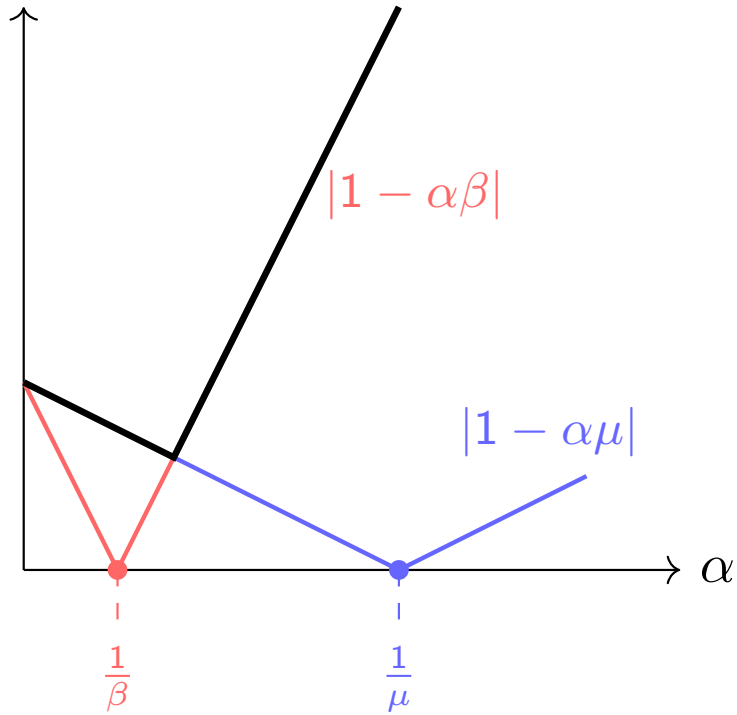
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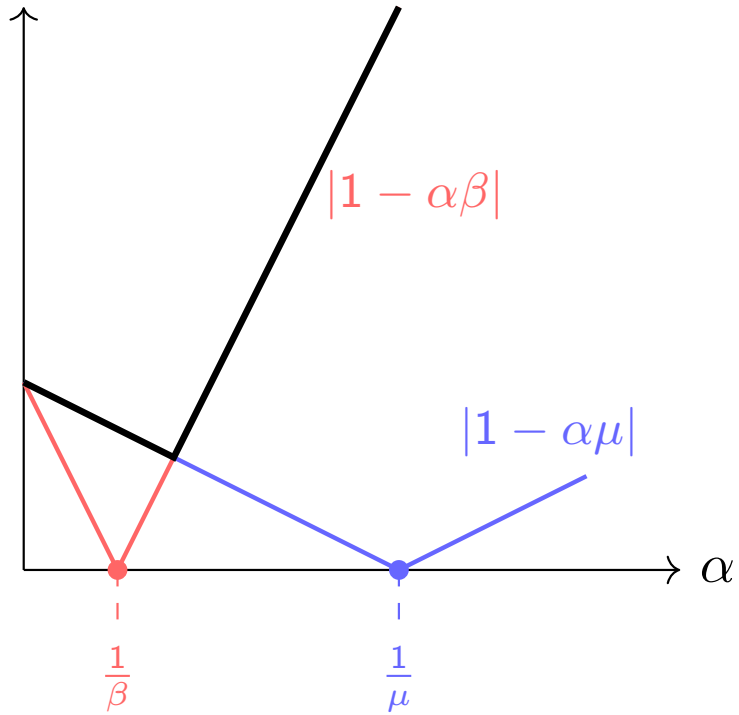
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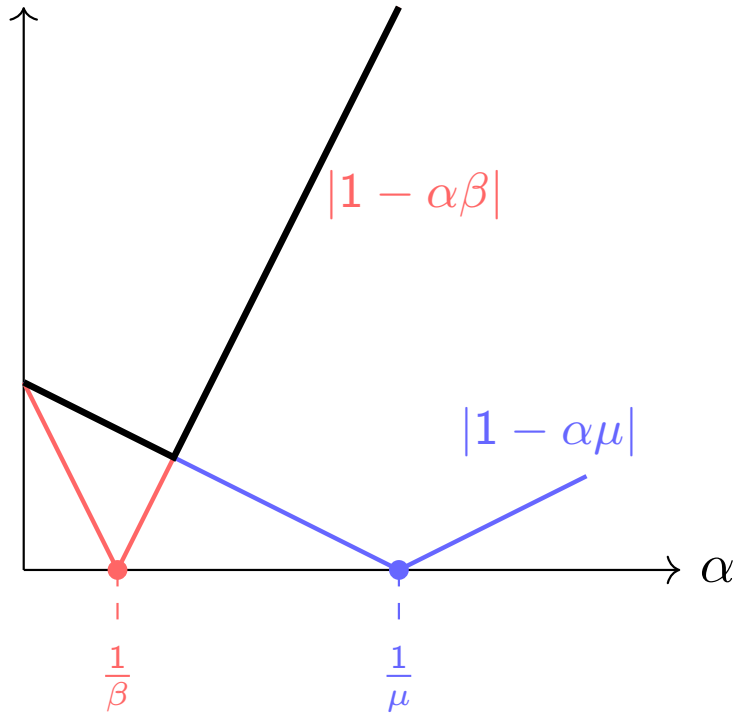
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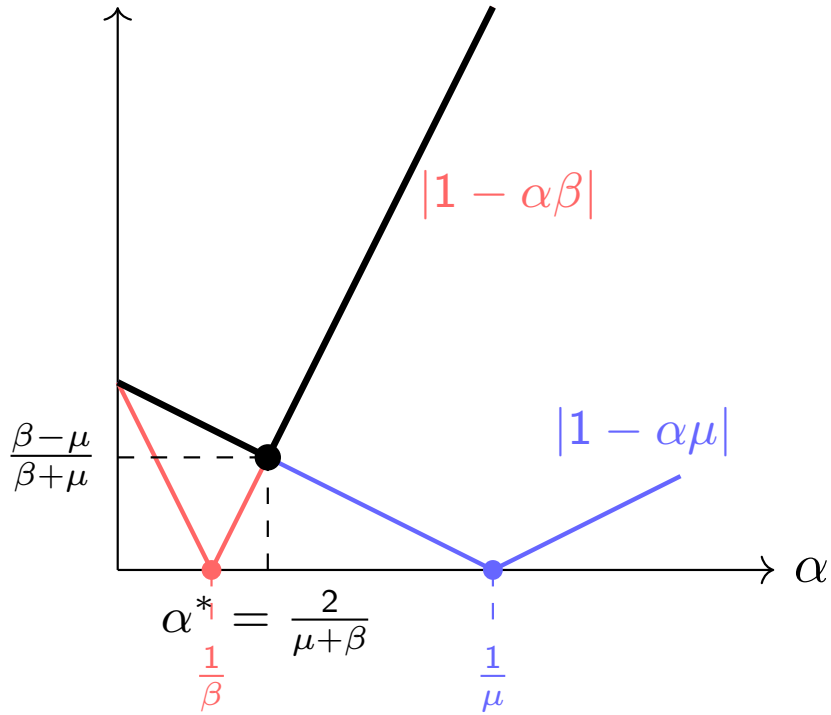


To minimize $q(\alpha)$, balance the two terms at their intersection:

$$1 - \alpha\mu = \alpha\beta - 1 \implies \alpha^* = \frac{2}{\mu + \beta}, \quad q(\alpha^*) = \frac{\beta - \mu}{\beta + \mu}$$

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However, $\frac{\kappa-1}{\kappa+1} < 1 - \frac{1}{\kappa}$ for $\kappa > 1$
So $\alpha^* = \frac{2}{\mu+\beta}$ is actually better than $\alpha = 1/\beta$, as we saw before.

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Theorem: GD with optimal step size $\alpha^* = \frac{2}{\mu+\beta}$ satisfies:

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For this function:

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The rate $(\frac{\kappa-1}{\kappa+1})^T$ is optimal for first-order methods on this function class (convex, smooth, κ-conditioned)

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Both rates are optimal for constant-step-size gradient descent!