Randomized Algorithms

Cut sparsification I: uniform edge sampling

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Fall 2026

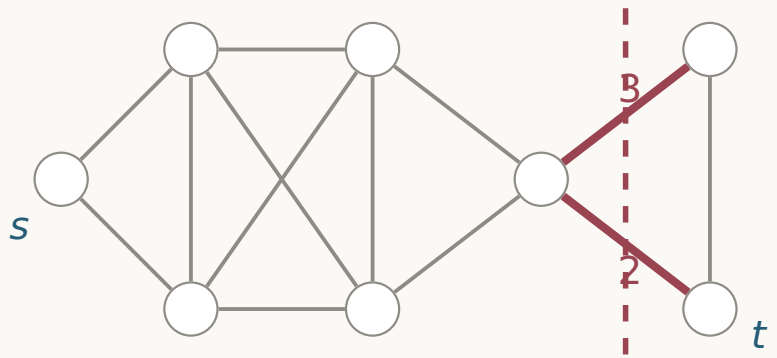
PART 1

The object we want

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A smaller network for a maximum-flow computation

A network has n vertices, $m \gg n$ edges, and a capacity $w_e \geq 0$ on each edge. We want to approximate the maximum flow between fixed terminals s and t.



Unlabeled edges have capacity 1. The dashed line shows one cut, not necessarily a minimum cut.

Could we replace the network by a much smaller weighted graph and solve the flow problem there? The original graph stays fixed; we are compressing it, not changing its capacities.

Maximum flow is a minimum over cuts

For fixed terminals s, t, let $\mathcal{C}_{s,t} = \{S \subset V : s \in S,\ t \notin S\}$. By max-flow/min-cut,

$$F_G(s,t) = \min_{S \in \mathcal{C}_{s,t}} c_G(S).$$

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Take the minimum over $S \in \mathcal{C}_{s,t}$ in each term:

$$\begin{aligned}(1-\varepsilon)\min_{S \in \mathcal{C}_{s,t}} c_G(S) &\le \min_{S \in \mathcal{C}_{s,t}} c_H(S) \\ &\le (1+\varepsilon)\min_{S \in \mathcal{C}_{s,t}} c_G(S).\end{aligned}$$

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Thus, for *every* choice of terminals in the same network,

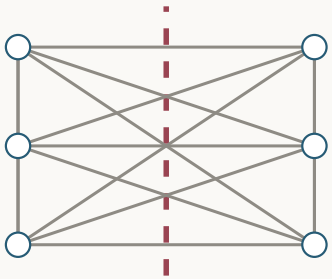
$$\boxed{(1-\varepsilon)F_G(s,t) \le F_H(s,t) \le (1+\varepsilon)F_G(s,t).}$$

The approximation is to the maximum-flow *value*; a flow on H is not automatically a feasible flow on G.

Why not keep a spanning tree?

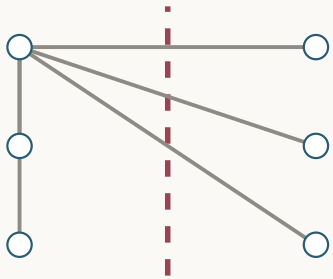
A spanning tree retains only $n - 1$ edges. But it can change the capacity of a cut substantially. Take the cut S separating three vertices of K_6 from the other three.

Clique K_6



$c_G(S) = 9$

Star spanning tree

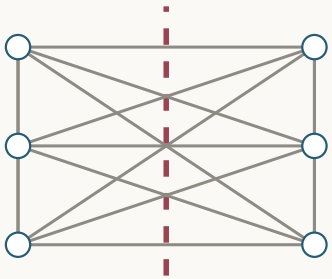


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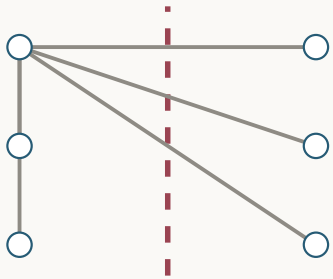
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The star keeps only three of the nine edges crossing this cut. We would like a sparse *weighted* graph that approximates every cut, not necessarily a tree.

Method	Number of edges	Construction
Spanning tree	$n - 1$	deterministic; poor cut approximation
Benczúr–Karger	$O(n \log n/\varepsilon^2)$	randomized sampling
BSS sparsifier	$O(n/\varepsilon^2)$	deterministic, stronger guarantee

What should a smaller graph preserve?

For a graph $G = (V, E, w)$ with nonnegative edge capacities, define

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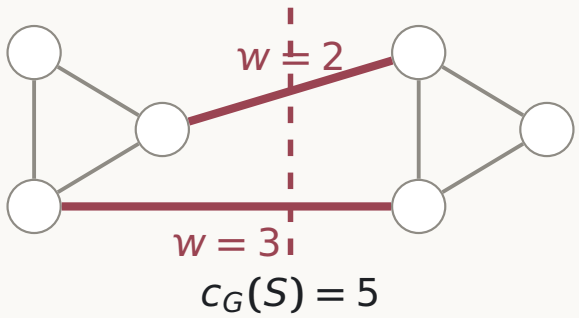
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There are $2^{n-1} - 1$ distinct nontrivial cuts. One graph H must approximate them all simultaneously.



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Thus unbiasedness holds by construction. An edge of capacity 2 sampled with probability $1/2$ is deleted or retained with capacity 4:

$$\mathbb{E}\widetilde{w}_e = \tfrac{1}{2}(0) + \tfrac{1}{2}(4) = 2.$$

By linearity, $\mathbb{E}c_H(S) = c_G(S)$ for each cut. But a realized sample need not be close to its expectation; we must control the random error for every cut at once.

PART 2

Warm-up: one probability for every edge

2

Uniform sampling: one probability for every edge

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$$Y_S = \sum_{i=1}^{c} X_i, \qquad X_i \overset{\text{iid}}{\sim} \text{Bernoulli}(p), \qquad Y_S \sim \text{Binomial}(c, p).$$

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The weight of the sampled cut is $c_H(S) = Y_S/p$, with $\mathbb{E}c_H(S) = c$. For example, if $c = 4$ and $p = 1/2$, the sampled value is $2Y_S$ and may be 0, 2, 4, 6, or 8. We want to bound the probability that it differs from 4 by more than a chosen relative error.

How accurate is one sampled cut?

Start with the failure event we wish to bound:

$$\begin{aligned}\Pr\big(|c_H(S) - c| > \varepsilon c\big) &= \Pr\left(\left|\frac{Y_S}{p} - c\right| > \varepsilon c\right) \\ &= \Pr\big(|Y_S - pc| > \varepsilon pc\big).\end{aligned}$$

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Since $Y_S \sim \mathrm{Binomial}(c, p)$, multiplicative Chernoff gives, for $0 < \varepsilon < 1$,

$$\boxed{\Pr\big(|c_H(S) - c| > \varepsilon c\big) \le 2\exp(-\varepsilon^2 pc/3).}$$

A cut with more crossing edges has a smaller relative-error probability, for the same p.

Choosing p to preserve every cut

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$$\begin{aligned}\Pr(\exists \text{ inaccurate cut}) &\leq \sum_{\text{distinct cuts } S} 2e^{-\varepsilon^2 p c_G(S)/3} \\ &\leq (2^{n-1}-1)2e^{-\varepsilon^2 p\lambda/3} \\ &< 2^n e^{-\varepsilon^2 p\lambda/3}.\end{aligned}$$

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To make this at most δ, it suffices that

$$\boxed{p\lambda \geq \frac{3(n\ln 2 + \ln(1/\delta))}{\varepsilon^2}.}$$

If the right-hand side exceeds λ, we must use $p = 1$, which means we just keep the original graph, unsparsified!

Why this bound gives no compression

For a simple unweighted graph, $\lambda \leq n - 1$. But the condition we just derived is

$$p\lambda \geq \frac{3(n \ln 2 + \ln(1/\delta))}{\varepsilon^2} > n - 1 \qquad (0 < \varepsilon < 1,\ \delta < 1).$$

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Thus the bound never certifies any sampling with $p < 1$. Where was the estimate wasteful? In the union bound, we replaced the actual cut size $c_G(S)$ by λ for *every* cut, although large cuts have exponentially smaller failure probabilities.

To improve the analysis, we need to count cuts according to their size.

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Rather than multiply the worst failure probability by the number of *all* cuts, we should count how many cuts have size at most $\alpha\lambda$ and apply the union bound separately to groups of cuts of similar sizes.

Question: How many cuts can have size at most $\alpha\lambda$?

PART 3

Repairing the uniform-sampling proof

3

How many cuts are near-minimum?

Write $\lambda = \mathrm{MINCUT}(G)$. For an integer $\alpha \geq 1$, consider

$$N_\alpha = \#\{\text{distinct cuts } S : c_G(S) \leq \alpha\lambda\}.$$

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Preserving a fixed cut during contraction

Fix C with $c_G(C) \leq \alpha\lambda$ and contract uniformly random edges until only 2α supervertices remain (assume $2\alpha < n$).

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Fix C with $c_G(C) \le a\lambda$ and contract uniformly random edges until only $2a$ supervertices remain (assume $2a < n$). Suppose r supervertices remain and no edge of C has been contracted. Every supervertex has degree at least λ, hence there are at least $r\lambda/2$ edges. At most $a\lambda$ of those edges cross C.

$$\Pr(\text{next contraction crosses } C \mid C \text{ survived}) \le \frac{a\lambda}{r\lambda/2} = \frac{2a}{r}.$$

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So the chance of avoiding C at this step is at least $(r - 2a)/r$.

Probability that the cut survives

Multiply the conditional probabilities from $r = n$ down to $r = 2a + 1$:

$$\Pr(C \text{ survives}) \geq \prod_{r=2a+1}^{n} \frac{r-2a}{r}$$
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This lower bound holds for *each* cut C of size at most $a\lambda$. We will use it to count how many such cuts there can be.

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Fix C with $c_G(C) \le a\lambda$. If it survives contraction, no supervertex contains vertices from both sides of C, so step 2 can select it. Thus

$$\Pr(\text{output} = C) \ge \underbrace{\frac{1}{\binom{n}{2a}}}_{\text{survives}} \underbrace{\frac{1}{2^{2a-1}-1}}_{\text{selected}} \ge \frac{1}{\binom{n}{2a}2^{2a-1}}.$$

Consequently $N_a \le \binom{n}{2a}2^{2a-1} \le n^{2a}$.

Karger's cut-counting theorem

Counting near-minimum cuts

For every integer $\alpha \geq 1$, the number of distinct cuts of an unweighted graph G with value at most $\alpha\lambda$ is at most $n^{2\alpha}$, where $\lambda = \text{MINCUT}(G)$.

For $2\alpha < n$, this follows from the contraction experiment. The last inequality uses

$$\binom{n}{2\alpha} \leq \frac{n^{2\alpha}}{(2\alpha)!} \quad \text{and} \quad (2\alpha)! \geq 2^{2\alpha-1}.$$

If $2\alpha \geq n$, the bound is immediate: there are only $2^{n-1} - 1 \leq n^{2\alpha}$ distinct cuts altogether.

Union bound over cuts of similar sizes

Let $\lambda = \text{MINCUT}(G)$. Group cuts into ranges

$$j\lambda \le c_G(S) < (j+1)\lambda, \qquad j = 1, 2, \ldots$$

There are at most $n^{2(j+1)}$ cuts in the jth range, and each fails with probability at most $2e^{-\varepsilon^2 pj\lambda/3}$.

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For example, with $p\lambda \ge 3(d+6)\ln n/\varepsilon^2$, the sum is at most

$$2n^2 \sum_{j\ge 1} n^{-(d+4)j} < n^{-d} \qquad (n \ge 2, d \ge 1).$$

The required $p\lambda$ has dropped from order n to order $\log n$.

Uniform sampling: what the better analysis proves

For $d \geq 1$, choose

$$p = \min\left\{1, \frac{3(d+6)\ln n}{\varepsilon^2 \lambda}\right\}, \qquad \lambda = \mathrm{MINCUT}(G).$$

If $p < 1$, the preceding calculation shows that *all* cuts are preserved within relative error ε with probability at least $1 - n^{-d}$. If $p = 1$, preservation is exact.

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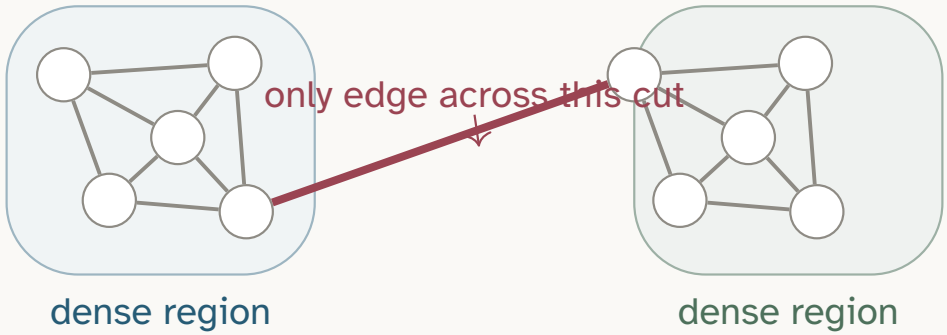
If $p < 1$, the preceding calculation shows that *all* cuts are preserved within relative error ε with probability at least $1 - n^{-d}$. If $p = 1$, preservation is exact. **How would we obtain λ?** We are assuming it is known while analyzing the sampler. A minimum-cut algorithm can compute it. The basic random contraction algorithm from the previous lecture can also find it with amplification, but is not a fast preprocessing method on large graphs.

A known lower bound on λ is safe in the denominator; an overestimate can cause undersampling.

The improved cut count makes the clique example possible. But the same global probability can still be wasteful for graphs with a small bottleneck.

A bridge forces uniform sampling to retain every edge

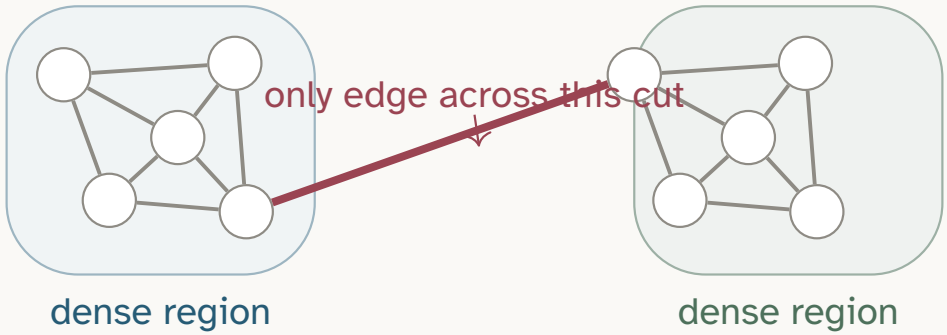
Connect two large cliques by a single bridge (only a few clique edges are drawn).



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Here $\mathrm{MINCUT}(G) = 1$. Our uniform-sampling guarantee then uses $p = 1$: the bridge forces us to retain every edge, including the many edges inside the cliques. The issue is the *global* choice of p: the bridge needs special care, but that does not mean all clique edges should be sampled with the same probability.

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3. **Preserve all cuts at once.** Unbiasedness is not enough: one sampled graph must approximate every cut with high probability.

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A cut with c crossing edges was easy to control when pc was sufficiently large. With a separate probability p_e, a retained edge contributes $1/p_e$ to any cut it crosses. For an edge that is *not* retained deterministically, a natural restriction is to prevent its single random contribution from being too large relative to a cut it crosses:

$$\frac{1}{p_e} \leq \frac{c_G(S)}{\rho} \qquad \text{for every cut } S \text{ crossed by } e,$$

where ρ is on the order of $\varepsilon^{-2} \log n$.

A useful restriction on an edge's sampling probability

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where ρ is on the order of $\varepsilon^{-2} \log n$. A cut with few edges imposes a stronger restriction than a large cut. This suggests choosing sampling probabilities according to how small a cut can contain an edge.

Next lecture: How can we measure that local connectivity, keep only $O(n \log n/\varepsilon^2)$ edges on average, and still prove that *all* cuts are accurate?