Homework 1

CSCI 4030 / CSCI 6220: Randomized Algorithms — Fall 2026

Rensselaer Polytechnic Institute — 100 points

Justify every claim. A numerical answer without an argument earns no credit. Acknowledge any collaborators and any external sources you consult, including AI tools, and describe clearly what role they played in your submission.

[5 points] Problem 1 (Comparisons involving one element in randomized Quicksort).

Randomized Quicksort is run on n distinct elements with ranks

$$1 < 2 < \cdots < n.$$

Recall that elements of ranks $i < j$ are compared with probability

$$\frac{2}{j - i + 1}.$$

Fix $k \in \{1, \ldots, n\}$. Determine the expected number of comparisons involving the element of rank k. Give an exact expression in terms of the harmonic numbers

$$H_m = \sum_{r=1}^{m} \frac{1}{r}, \qquad H_0 = 0.$$

Then determine which ranks k minimize this expectation and which ranks maximize it. Prove your answers.

[10 points] Problem 2 (Inversions corrected by Bubblesort).

Mitzenmacher & Upfal, Exercise 2.22.

Let $a_1, a_2, \ldots, a_n$ be a list of n distinct numbers. The pair (a_i, a_j) is *inverted* if $i < j$ but $a_i > a_j$. The Bubblesort algorithm repeatedly swaps adjacent inverted elements until no inversions remain:

for $i = 1, 2, \ldots, n-1$:
  for $j = n, n-1, \ldots, i+1$:
    if $a_{j-1} > a_j$:
      swap a_{j-1} and a_j

During outer iteration i, the inner loop moves the smallest element of $A[i..n]$ leftward until it reaches position i. Thus, after iteration i, positions $1, \ldots, i$ contain the i smallest elements in sorted order.

For example, suppose the initial array is $[4, 2, 5, 1, 3]$. The table records the array at the beginning and after each outer iteration. In every completed row, the bold entries form the sorted prefix guaranteed by the invariant above.

Outer iteration	Array after the iteration
Initial	$[4, 2, 5, 1, 3]$
$i = 1$	$[\mathbf{1}, 4, 2, 5, 3]$
$i = 2$	$[\mathbf{1}, \mathbf{2}, 4, 3, 5]$
$i = 3$	$[\mathbf{1}, \mathbf{2}, \mathbf{3}, 4, 5]$
$i = 4$	$[\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, 5]$

After iteration i, the bold prefix contains the i smallest input elements in sorted order. The remaining suffix need not yet be sorted.

Suppose the input is a uniformly random permutation, equally likely to be any of the $n!$ orderings of n distinct numbers. Determine the expected number of swaps performed by Bubblesort. As part of your argument, explain how swapping two adjacent inverted elements changes the number of inversions.

[10 points] Problem 3 (Wald's identity for an independent number of summands).

Adapted from Jeff Erickson's lecture on discrete probability.

Let N be a nonnegative integer-valued random variable with $\mathbb{E}[N] < \infty$. Let

$$X_1, X_2, \ldots$$

be identically distributed integrable random variables, and assume that the sequence $(X_i)_{i \geq 1}$ is independent of N. Define

$$S_N = \sum_{i=1}^{N} X_i,$$

where $S_0 = 0$. Let $\mu = \mathbb{E}[X_1]$. The random variables X_i need not be mutually independent.

(a) (3 points) Fix $n \geq 0$. Use the independence of N and $X_1, X_2, \ldots$ to show that

$$\mathbb{E}[S_N \mid N = n] = n\mu.$$

(b) (3 points) Condition on N to prove that

$$\mathbb{E}[S_N] = \mathbb{E}[N]\,\mathbb{E}[X_1].$$

(c) (4 points) Explain where independence from N was used in your proof. Give an example showing that the identity can fail when N is allowed to depend on the X_i.

This result is the independent-index form of Wald's identity.

[10 points] Problem 4 (Correctness of a shuffle).

The following algorithm shuffles an array $A[1..n]$ of n distinct elements in place.

for $i = n, n-1, \ldots, 2$:
    draw j uniformly from $\{1, 2, \ldots, i\}$, independent of all earlier draws
    swap $A[i]$ and $A[j]$

Prove that the algorithm is correct in the following sense: for every one of the $n!$ possible orderings σ of the n input elements,

$$\mathbb{P}[\,\text{the final contents of } A \text{ equal } \sigma\,] = \frac{1}{n!}.$$

[10 points] Problem 5 (Reading a corrupted lookup table).

Mitzenmacher & Upfal, Exercise 1.18.

Let $F : \{0, \ldots, n-1\} \to \{0, \ldots, m-1\}$ satisfy

$$F\big((x+y) \bmod n\big) = \big(F(x) + F(y)\big) \bmod m \qquad \text{for all } 0 \le x, y \le n-1.$$

The only way to evaluate F is through a lookup table T. Assume that 5 divides n. An adversary has altered exactly $n/5$ of the n entries of T, so $T(x) = F(x)$ for at least $4n/5$ values of x, but we do not know which entries were altered. The adversary fixed the corrupted set in advance, without seeing any of our random choices.

Give a randomized algorithm that, on every input $z \in \{0, \ldots, n-1\}$, outputs $F(z)$ with probability at least $1/2$, using only two table lookups and $O(1)$ additional arithmetic operations. Prove the guarantee. Your bound must hold for every z, including those z for which $T(z)$ itself is corrupted; explain why your argument does not depend on which entries the adversary chose.

[15 points] Problem 6 (A reinforcing urn).

Mitzenmacher & Upfal, Exercise 1.6.

Fix an integer $n \ge 2$. A bin initially contains one black ball and one white ball. We repeat the following step: choose one ball from the bin uniformly at random, then return it to the bin together with one additional ball of the same colour. We stop when the bin contains n balls.

Let W_n denote the number of white balls present when the bin contains n balls. Prove that W_n is uniformly distributed on $\{1, 2, \ldots, n-1\}$; that is,

$$\mathbb{P}[W_n = j] = \frac{1}{n-1} \qquad \text{for every } j \in \{1, \ldots, n-1\}.$$

[15 points] Problem 7 (The longest run of heads).

(a) (6 points) A fair coin is flipped n times. For an integer $k \ge 1$, give an upper bound on the probability that the sequence contains a run of at least $\lceil \log_2 n \rceil + k$ consecutive heads. Your bound should decay exponentially in k and should not depend on n.

(b) (9 points) A fair coin is flipped n times. Show that for sufficiently large n, the probability that the sequence contains *no* run of length at least $\lceil \log_2 n - 2 \log_2 \log_2 n \rceil$ is less than $1/n$.

[25 points] Problem 8 (Rosencrantz and Guildenstern).

Adapted from Mitzenmacher & Upfal, Exercises 1.9 and 2.16(b).

On their long journey from Denmark to England, Rosencrantz and Guildenstern amuse themselves with the following game using a fair coin.

First, Rosencrantz flips the coin repeatedly until it comes up tails. Then Guildenstern flips the coin repeatedly until he obtains as many consecutive heads as Rosencrantz obtained before his tail.

Here are three possible games:

	Rosencrantz	Guildenstern
1	`HHT`	`HTHH`
2	`T`	no flips
3	`HHHT`	`THHTHHTHTTHHH`

Guildenstern's turn ends as soon as the required run of heads first occurs. If Rosencrantz obtains no heads, Guildenstern makes no flips.

(a) (5 points) What is the expected number of flips in one of Rosencrantz's turns?

(b) (8 points) Suppose Rosencrantz obtains exactly k heads before his tail, where $k \geq 0$ is an integer. What is the expected number of flips in Guildenstern's ensuing turn? Your answer should be an exact function of k.

(c) (12 points) What is the expected total number of flips made by both players in one game? Your answer may be infinite.

Prove each of your answers.