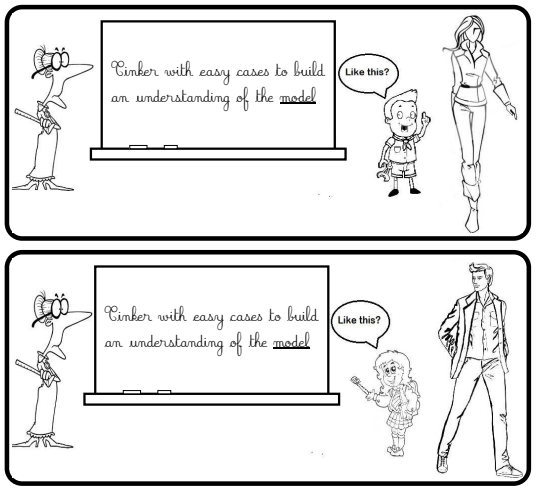
# Foundations of Computer Science
# Lecture 2

## Discrete Objects and Proof

The Cast of Discrete Objects
Some Basic Proofs



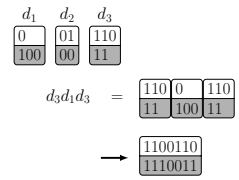

(Niteesh Thangaraj, RPI Class of 2020)

## Last Time

A taste of discrete math and computing (ebola, speed dating, friendship networks)

**$100**
*Distinct* subsets with the same sum.

**$1,000**
Domino Program




**Goal:** Want same top and bottom.

**Domino program:**
Input: dominos
Output: sequence that works
or
say it can't be done

**$10**
Create the best 'math'-cartoon.

Create a cartoon to illustrate/make fun of some discrete math you learned in this class.

If you submit one, I can use it in the future



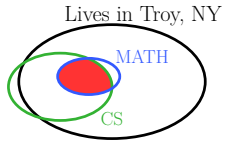
## Today: Discrete Objects and Proof

1. Discrete Objects
   - Sets
   - Sequences
   - Graphs
2. Proof
   - In 4 rounds of the speed-dating app, no one meets more than 12 people.
   - $x^2$ is even "is the same as" $x$ is even
   - Among *any* 6 people is a 3-clique or 3-war.
   - **Axioms.** The Well Ordering Principle.
   - $\sqrt{2}$ is not rational.



## Sets

1. Collection of objects, order does not matter: $F = \{f, o, x\}$; $V = \{a, e, i, o, u\}$.
   $F \cap V = \{o\}$ $\quad F \cup V = \{a, e, f, i, o, u, x\}$ $\quad \overline{F} = ?$
2. natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$ What is "…?"
   integers $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, \pm5, \ldots\}$
3. $E = \{2, 4, 6, 8, 10, 12, \ldots\}$ $\quad E' = \{2, 4, 6, 8, 10, 13, \ldots\}$ What is "…?"
4. $E = \{n \mid n = 2k;\ k \in \mathbb{N}\}$ ← no "…" **Pop Quiz:** Define $O = \{\text{odd numbers}\}$.
5. Rational numbers $\mathbb{Q} = \left\{r \,\middle|\, r = \frac{a}{b};\ a \in \mathbb{Z},\ b \in \mathbb{N}\right\}$
6. Subset $A \subseteq B$ (every element of $A$ is in $B$). $\varnothing \subseteq A$ for any $A$.
   Power set $\mathcal{P}(A) = \{\text{all subsets of } A\}$ **Pop Quiz:** $A = \{a, b\}$. What is $\mathcal{P}(A)$?
7. Set equality, $A = B$ means $A \subseteq B$ and $B \subseteq A$.
8. Set operations: Intersection, $A \cap B$
   Union, $A \cup B$
   Complement, $\overline{A}$
9. Venn Diagrams are a convenient way to represent sets.



## Sequences

- List of objects: order and repetition matter.

$$tap \neq taap \neq atp$$

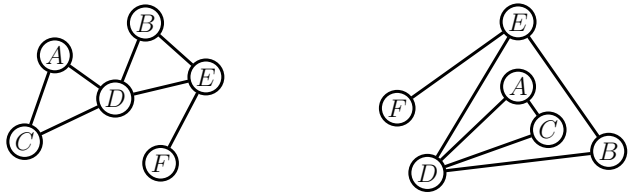
- We are mostly concerned with *binary sequences* composed of *bits* (ASCII code).

| $t$ | $a$ | $p$ |
|---|---|---|
| 01110100 | 01100001 | 01110000 |



## Graphs

Friendships between Alice, Bob, Charles, David, Edward, Fiona:




$$V = \{A, B, C, D, E, F\}.$$

$$E = \{(A, C), (A, D), (C, D), (B, D), (B, E), (D, E), (E, F)\}.$$

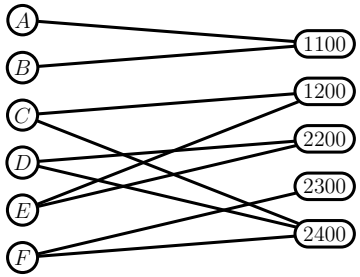
What matters is:
who the people are, that is the set $V$ of objects; and,
who is friends with whom, that is the set $E$ of relationships.

The picture with circles and links is a convenient *visualization* of the graph.



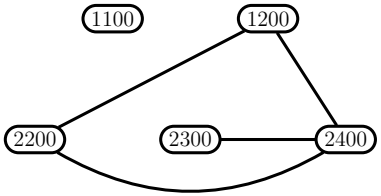
## Graphs and Different Types of Relationships

Affiliation graphs




Students and their courses.

Conflict graphs




Courses with students in common conflict. (Why?)



## Proof

It is Human to seek verification – proof.

- The sun has risen every morning in history. (*inductive proof*)

- In the speed dating ritual, no-one meets more than 12 people.

  *deductive proof:*
  In any round a person meets *at most* 3 new people. (Why?)
  There are 4 rounds, *ergo* at most $4 \times 3 = 12$ people can be met.

  Do you have any doubts? That is the beauty of deductive proof.

## When is a Number a Square

**Tinker!**

| $n$ | 0 | $\pm1$ | $\pm2$ | $\pm3$ | $\pm4$ | $\pm5$ | $\pm6$ | $\pm7$ | $\pm8$ | $\pm9$ | $\pm10$ | $\pm11$ | $\dots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n^2$ | **0** | 1 | **4** | 9 | **16** | 25 | **36** | 49 | **64** | 81 | **100** | 121 | $\dots$ |

**Conjecture.**
Even squares come from even numbers and even numbers have even squares.

*Proof.* (How do I convince you this is true, *without a doubt*?) Let's look at the *cases*

1. $n$ is even $\rightarrow\ n = 2k\ \rightarrow\ n^2 = 2(2k^2)\ \rightarrow\ n^2$ **is even**.
2. $n$ is odd $\rightarrow\ n = 2k+1\ \rightarrow\ n^2 = 2(2k^2+2k)+1\ \rightarrow\ n^2$ **is odd**.

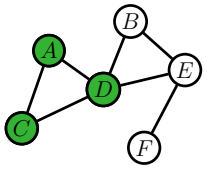
$n$ must be even or odd, and we made no assumptions about $n$ ($n$ is *general*).

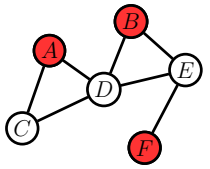
Are you convinced? ■

**Theorem.**
*Every* even square came from an even number and *every* even number has an even square.



## 3-war or 3-clique



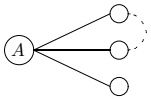




friend clique
war

**Theorem.**
Any 6-person friend network, has a 3-person friend clique or a 3-person war (or both).

*Proof.* For a *general* network with 6 people, there are two cases:

(i) $A$ has more friends than enemies.




Two friends are linked → 3-clique.
None are linked → 3-war.

(ii) $A$ has more enemies than friends.


Two friends are enemies → 3-war.
None are enemies → 3-clique. ■



## We Can't Prove Everything

- **Axioms:** A self-evident statement that is asserted as true without proof.
- **Conjectures:** A claim that is believed true but is not true until proven so.
- **Theorems:** A proven truth. You can take it to the bank.

**Axiom. The Well-Ordering Principle**
*Any* non-empty subset of $\mathbb{N}$ has a minimum element.
$\{2, 5, 4, 11, 7, 296, 81\}$; or,
$\{6, 19, 24, 18, \dots\}$.

**Exercises.**

- Construct a subset of $\mathbb{Z}$ (integers) that has no minimum element.
- Construct a positive subset of $\mathbb{Q}$ (rationals) that has no minimum element.



## A Gift from Hipassus: $\sqrt{2}$ is Irrational

**It may not be so.**

In which case $\sqrt{2}$ is rational,

$$\boxed{\sqrt{2}} = \left\{\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \frac{a_4}{b_4}, \dots\right\}$$ ← all possible ways to write $\sqrt{2}$ as a fraction

where $a_1, a_2, \dots$ are all integers and $b_1, b_2, \dots$ are all natural numbers.

Well ordering principle: there is a minimum $b_i$, call it $b_*$.

$\sqrt{2} = a_*/b_*$ and $a_*$ **and** $b_*$ **have no factor in common.** ($b_*$ is the minimum possible)

$$\sqrt{2} = \frac{a_*}{b_*} \quad \rightarrow \quad a_*^2 = 2b_*^2 \quad \rightarrow \quad a_* \text{ is even (why?).}$$

So, $a_* = 2k$ and

$$4k^2 = 2b_*^2 \quad \rightarrow \quad b_*^2 = 2k^2 \quad \rightarrow \quad b_* \text{ is even (why?).}$$

So, $a_*$ **and** $b_*$ **have the factor 2 in common.**

Reductio ad Absurdum

**FISHY!**

**It *must* be so!**

## A Proof Must Convince

A proof strings together "truths" to *convince* the reader of something *new*.

Our proof that $\sqrt{2}$ is irrational strung together several "truths":

- The well ordering principle.
- High-school algebra for manipulating equalities.
- Our Theorem on when a square is even.

**A proof's goal is always, always, ALWAYS to convince a reader of something.**



## Making and Proving Claim

**Three Steps for Making and Proving a Claim**

**Step 1: Precisely state the right thing to prove.** Often, creativity and imagination are needed. The claim should be non-trivial, i.e. useful, but also "provable" given the tools you have. Most importantly, the claim should be true (and how do you know that).

**Step 2: Prove the claim.** Sometimes a simple "genius" idea may be needed. Again, creativity and imagination play a role. Sometimes standard proof techniques can be used; you can become proficient in these techniques through training and practice.

**Step 3: Check the proof for correctness.** No creativity is needed to look a proof in the eye and determine if it is correct; to determine if you are convinced. Become an expert at this task. Don't allow anyone to claim bogus things and "convince" you with invalid proofs.

**Next.** How to make precise claims.