Homework 8

IMPORTANT NOTE:
Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.

Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 15.5. [A good warm-up for 15.4.]
(2) DMC Problem 15.9.
(3) DMC Problem 15.10 (a), (b).
(4) DMC Problem 15.40 (c), (d). [Note: You may assume Ω is finite.]
(5) DMC Problem 16.9.
(6) DMC Problem 16.22.
(7) DMC Problem 16.82 (a), (b). [A good warm-up problem.]

Problems to submit (50 points)

Show your work and explain your reasoning. A correct answer without adequate justification may receive little or no credit. As a reminder, the homework instructions require you to typeset your solutions, submit a pdf, and start each problem on a new page. Write in complete, grammatically sound sentences. The number in square brackets is the number of points each problem is worth.

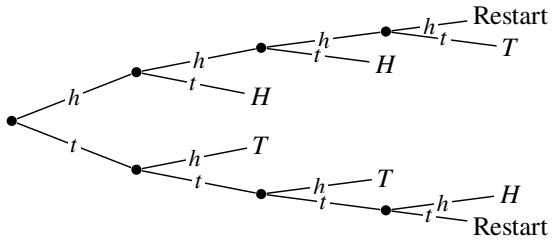
(1) **[8]** DMC Problem 15.4.
(2) **[8]** DMC Problem 15.8. [Note: In (a), "one 6" means "exactly one 6."]
(3) **[8]** DMC Problem 15.10 (c), (d).
(4) **[12]** DMC Problem 15.40 (e), (g). [6 points each] [Note: You may assume Ω is finite.]
(5) **[6]** DMC Problem 16.27.
(6) **[8]** DMC Problem 16.85. [Hint: Let w denote the probability of winning. Write an equation in terms of w.]

Bonus challenge: A better fair coin (up to 10 bonus points)

This problem is completely optional and is meant to be substantially more challenging than the required homework. Each part is graded all-or-nothing: a complete proof earns the listed bonus points, while an incomplete proof earns no points for that part. Bonus points are added to your homework score, up to a maximum score of **60/50**.

In class (and in Example 16.3 in the DMC book), we learned a procedure for the following objective. A biased coin gives h with probability $0 < p < 1$, and t with probability $q = 1 - p$, for some unknown value of p. Its tosses are independent. We want a procedure that stops with probability 1 and outputs H or T, each with probability $1/2$.[1] We want this procedure to work for *every* possible p, without the algorithm knowing p. The procedure we learned is due to von Neumann. We start tossing coins and look at the first two tosses. On ht, we stop and output H; on th we get T; on hh or tt, we re-start the procedure from the beginning.

Consider the alternative procedure A given by the tree below (adapted from DMC 16.77).



(1) **[3 bonus points]** Prove that A stops with probability 1 and gives a fair output.

(2) **[3 bonus points]** Let N denote von Neumann's procedure (from class, and Example 16.3 in DMC). Imagine applying N and A to the same sequence of tosses. Prove that whenever von Neumann's procedure has finished, A has also finished. Explain why, for every positive integer n, A is at least as likely to finish within n tosses. Calculate the probability that each procedure finishes within four tosses, and show that A has a strictly greater probability for every $0 < p < 1$.

(3) **[4 bonus points]** Construct a new procedure B for generating a fair coin toss such that: (1) for all n, B is at least as likely as A to terminate in $\leq n$ tosses, and (2) B is strictly more likely than A to finish within eight tosses. Prove your answer is correct.

[Hint: You can construct a new procedure that starts with a pass of A; if it restarts, remember its four tosses and run another pass of A. Can we find a way to remove some of the 'Restart' nodes?]

[1] Note that lowercase letters t and h label the input tosses from the biased coin; uppercase letters T and H are our output tosses (which we must make equally likely).