# Homework 7

**IMPORTANT NOTE:**
**Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.**
This homework covers Chapters 13 and 14.

## Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 13.25 (a).
(2) DMC Problem 13.29 (s).
(3) DMC Problem 13.52 (a), (b). [A good warm-up for 13.52 (c) and 14.32.]
(4) DMC Problem 13.53 (j). [A good warm-up for 13.84.]
(5) DMC Problem 13.64 (a), (b).
(6) DMC Problem 14.1 (c).
(7) DMC Problem 14.2 (b).
(8) DMC Problem 14.17 (a). [A good warm-up for 14.17 (b).]
(9) DMC Problem 14.49. [A good warm-up for 14.50 and 14.52.]

## Problems to submit (100 points)

**Show your work and explain your reasoning.** A correct answer without adequate justification may receive little or no credit. Write in complete, grammatically sound sentences and submit legible work. The number in square brackets is the number of points each problem is worth. Exact expressions involving factorials, powers, or binomial coefficients are acceptable.

(1) **[4]** DMC Problem 13.25 (b).
(2) **[4]** DMC Problem 13.31.
(3) **[8]** DMC Problem 13.34. [Note: The teams are unlabeled.]
(4) **[6]** DMC Problem 13.52 (c).
(5) **[8]** DMC Problem 13.84. [Note: The $z_i$ are integers. Hint: Consider successive differences.]
(6) **[10]** DMC Problem 13.88 (b).
(7) **[8]** DMC Problem 13.99 (a), (b).
(8) **[10]** DMC Problem 13.99 (c).
(9) **[10]** DMC Problem 14.17 (b).
(10) **[6]** DMC Problem 14.22 (b).
(11) **[8]** DMC Problem 14.32.
(12) **[10]** DMC Problem 14.44.
(13) **[4]** DMC Problems 14.50. [Note: The numbers are integers.]
(14) **[4]** DMC Problems 14.52. [Note: The numbers are integers.]

## Bonus: What else can Prüfer codes tell us?

This problem is completely optional and is meant to be more challenging than the required homework. Each part is graded all-or-nothing: a complete proof earns the listed bonus points; an incomplete proof earns no points for that part. Bonus points are added to your homework score, up to a maximum of **110/100**.

In DMC Problem 13.99, you encountered Prüfer codes and used them to establish *Cayley's formula*: that there are exactly $n^{n-2}$ trees on $n$ labeled vertices. This is a remarkable result! By way of comparison, there is no known closed formula for the number of trees on $n$ unlabeled vertices.

Prüfer codes establish a bijection between labeled trees and sequences. This bijection has other uses: it lets us answer questions about potentially-complicated graph structures by using counting arguments. This can be useful in algorithms for representing and generating trees, and in counting network topologies subject to degree constraints (e.g., if a communication network requires certain vertices to have specified numbers of connections).

Let $T$ be a tree on vertex set $\{1, \ldots, n\}$, where $n \geq 3$. Construct its code $P(T)$ as follows: repeatedly delete the leaf with the smallest label and record the label of its neighbor, *stopping when two vertices remain*. Thus $P(T)$ has length $n - 2$. Equivalently, take the code in DMC Problem 13.99 and delete its final entry (which is always $n$).

(a) **[3 bonus points] Degrees and multiplicities.** Prove that each vertex $v$ appears exactly $\deg_T(v) - 1$ times in $P(T)$. Deduce that $v$ is a leaf if and only if it does not appear in the code.

(b) **[4 bonus points] Trees with prescribed degrees.** Let $d_1, \ldots, d_n$ be positive integers satisfying

$$\sum_{i=1}^{n} d_i = 2(n-1).$$

Determine, with justification, the number of labeled trees $T$ for which $\deg_T(i) = d_i$ for every $i \in \{1, \ldots, n\}$.

(c) **[3 bonus points] Paths and stars.** Use Prüfer codes to determine, with justification, the number of labeled trees on $n$ vertices that are (i) paths and (ii) stars.