Homework 2

IMPORTANT NOTE:
Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.

Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 4.3
(2) DMC Problem 4.4
(3) DMC Problem 4.9 (a)
(4) DMC Problem 4.10 (g). Note: The statement is not correct as written for all $n \in \mathbb{Z}$. What should we change about the statement to make it true?
(5) DMC Problem 4.27
(6) DMC Problem 4.37 (a) and 4.38 (a)
(7) DMC Problem 4.47. (Humans have known some version of this proof since ca. 300 BC!)
(8) DMC Problem 5.3 (b), (c)
(9) DMC Problem 5.10 (d). You may assume $n \in \{1, 2, \ldots\}$.
(10) Give a brief argument showing the following equivalences:

(a) $\neg p \overset{\text{eqv}}{\equiv} (p \to \mathtt{F})$
(b) $p \vee q \overset{\text{eqv}}{\equiv} (p \to \mathtt{F}) \to q$
(c) $p \wedge q \overset{\text{eqv}}{\equiv} \big(p \to (q \to \mathtt{F})\big) \to \mathtt{F}$
(d) $(a \wedge b) \to c \overset{\text{eqv}}{\equiv} a \to (b \to c)$
(e) $(a \vee b) \to c \overset{\text{eqv}}{\equiv} (a \to c) \wedge (b \to c)$

Can you think of intuitive interpretations of (any or all of) these equivalences?

Problems to submit (100 points)

Show your work and explain your reasoning. A correct answer without adequate justification may receive little or no credit. Write in complete, grammatically sound sentences and submit legible work. The number in square brackets is the number of points each problem is worth.

(1) **[5]** DMC Problem 4.5 (a)
(2) **[10]** DMC Problem 4.7 (b) and (c)
(3) **[15]** DMC Problem 4.10 (d), (m), and (n).
(4) **[10]** DMC Problem 4.26
(5) **[8]** DMC Problem 4.37 (d), (j)
(6) **[8]** DMC Problem 4.38 (d), (j)
(7) **[10]** DMC Problem 5.10 (k), (o). You may assume $n \in \{1, 2, \ldots\}$.
(8) **[12]** DMC Problem 5.17
(9) **[12]** DMC Problem 5.75 (a)
(10) **[10 points total]** Recall that a compound proposition is a formula with variables such as $p, q, r, \ldots$ and operations $\neg, \wedge, \vee, \to$. A compound proposition is a *tautology* if it is true (T) for *every* possible

assignment of values (T or F) to the variables of the proposition. Give a brief argument for why each of the following are tautologies.

(a) **[2]** $p \vee (\neg p)$

(b) **[2]** $\big(p \wedge (p \to q)\big) \to q$

(c) **[2]** $\neg\neg p \to p$

(d) **[2]** $p \to \neg\neg p$

(e) **[2]** $(p \to q) \to (\neg q \to \neg p)$

Bonus challenge: A Different Form of Logic (up to 10 bonus points)

This bonus problem is completely optional and is meant to investigate interesting computer science material outside the usual scope of the course. Each part is graded all-or-nothing: a complete proof earns the listed bonus points, while an incomplete proof earns no points for that part. Bonus points are added to your homework score, up to a maximum score of **110/100**.

Standard logic is *two-valued*, i.e., all propositions are either true (written T) or false (written F). Sometimes, it is useful to use alternative versions of logic that model things like uncertainty or computational resources. Such logics have practical applications in software or hardware verification.

For this problem, we define *three-valued logic* to be a logical system in which propositions can have three values, T, L, and F. We interpret T and F as usual, but L models statements that become true over time, i.e., they are not true now but they will become true **L**ater (say, tomorrow). We define the operators $\vee, \wedge, \to, \neg$ in three-valued logic as follows:

$x \vee y$	$y = \mathtt{F}$	$y = \mathtt{L}$	$y = \mathtt{T}$
$x = \mathtt{F}$	F	L	T
$x = \mathtt{L}$	L	L	T
$x = \mathtt{T}$	T	T	T

$x \wedge y$	$y = \mathtt{F}$	$y = \mathtt{L}$	$y = \mathtt{T}$
$x = \mathtt{F}$	F	F	F
$x = \mathtt{L}$	F	L	L
$x = \mathtt{T}$	F	L	T

$x \to y$	$y = \mathtt{F}$	$y = \mathtt{L}$	$y = \mathtt{T}$
$x = \mathtt{F}$	T	T	T
$x = \mathtt{L}$	F	T	T
$x = \mathtt{T}$	F	L	T

x	$\neg x$
F	T
L	F
T	F

Note that we intentionally do not model statements that are true now, but become false later; this is part of the reason we define $\neg\mathtt{L} = \mathtt{F}$ and $\mathtt{L} \to \mathtt{F} = \mathtt{F}$. Similar to standard logic, we consider a compound proposition in our three-valued logic to be a tautology if it equals T for *every* possible assignment of values T, L, or F to the variables of the proposition.

(I) For **[1 bonus point each]**, determine which of the propositions in the above Homework Problem (10) are tautologies in three-valued logic. Label the five parts of this bonus problem (a) through (e) as above, and justify all your answers.

(II) For **[5 bonus points]**, prove that if a compound proposition P is a tautology in standard (two-valued) logic, then for every possible assignment of values T, L, or F to the variables of the proposition, the proposition P has value T or L in three-valued logic.
[Hint: A useful tool might be induction on the number of occurrences of operators "¬", "∨", "∧", or "→" in P. You might also want to phrase your proof in terms of some function $f : \{\mathtt{T}, \mathtt{L}, \mathtt{F}\} \to \{\mathtt{T}, \mathtt{F}\}$, where $f(t)$ "approximates" the meaning of t. We can set $f(\mathtt{T}) = \mathtt{T}$ and $f(\mathtt{F}) = \mathtt{F}$, but what do you think $f(\mathtt{L})$ should be? Don't be afraid to prove something more general than the stated theorem!**]**