Homework 11

IMPORTANT NOTE:
Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.

Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 21.7
(2) DMC Problem 21.11 (a), (b)
(3) DMC Problem 21.23 (b), (c)
(4) DMC Problem 21.45
(5) DMC Problem 21.55
(6) DMC Problem 22.5
(7) DMC Problem 22.11 (a), (c), (e)
(8) DMC Problem 22.25 (c), (d)
(9) DMC Problem 22.27 (a)
(10) DMC Problem 22.32

Problems to submit (100 points)

Show your work and explain your reasoning. A correct answer without adequate justification may receive little or no credit. Write in complete, grammatically sound sentences and submit legible work. The number in square brackets is the number of points each problem is worth.

(1) **[5]** DMC Problem 21.16 [Assume $n \geq 1$ and $0 < p < 1$.]
(2) **[15]** DMC Problem 21.24 [Interpret the second part of (c) as "When is the color of the *two draws* independent?"]
(3) **[10]** DMC Problem 21.35 [Assume both variances are positive.]
(4) **[15]** DMC Problem 21.39 (a), (b)
(5) **[10]** DMC Problem 21.59

    You may use a calculator or software to evaluate the exact binomial sum. You must derive that sum and the Chebyshev bound, and compare the results.
(6) **[10]** DMC Problem 22.11 (b), (d), (f)
(7) **[5]** DMC Problem 22.13 (b)
(8) **[10]** DMC Problem 22.23 (c)
(9) **[10]** DMC Problem 22.29
(10) **[10]** DMC Problem 22.35

Bonus: The Cantor set

This problem is completely optional and worth up to **10 bonus points**, for a maximum homework score of **110/100**. It is DMC Problem 22.40, restated below to clarify the treatment of endpoints and ternary expansions.

Start with $C_0 = [0, 1]$. To obtain C_{k+1}, remove the open middle third of each interval in C_k, keeping all endpoints. For example,

$$C_1 = [0, 1/3] \cup [2/3, 1], \qquad C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1].$$

The *Cantor set* consists of the points that are never removed:

$$C = \bigcap_{k=0}^{\infty} C_k.$$

(a) **[2 bonus points]** Show that C_k consists of 2^k disjoint intervals, each of length 3^{-k}. Find their total length and show that it tends to zero as $k \to \infty$.

(b) **[3 bonus points]** A ternary (base-three) expansion is an expression

$$x = 0.t_1t_2t_3\cdots_3 = \sum_{i=1}^{\infty} \frac{t_i}{3^i}, \qquad t_i \in \{0, 1, 2\}.$$

Show that $x \in C_k$ if and only if x has a ternary expansion whose first k digits belong to $\{0, 2\}$. Then show that $x \in C$ if and only if x has a ternary expansion using only the digits 0 and 2.

Some numbers have two ternary expansions; for example, $1/3 = 0.1000\cdots_3 = 0.0222\cdots_3$. The statements above require the *existence* of an appropriate expansion. Include $1 = 0.2222\cdots_3$ when describing points in $[0, 1]$.

(c) **[5 bonus points]** Prove that C is uncountable. Your proof must justify why the correspondence you use between sequences and points has the properties you need.

You have shown that an uncountable set can remain even though the total length of the intervals covering it tends to zero!