Homework 10

IMPORTANT NOTE:
Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.

Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 19.9
(2) DMC Problem 19.35
(3) DMC Problem 20.4
(4) DMC Problem 20.18
(5) DMC Problem 20.26 (a)(i)
(6) DMC Problem 20.45

Problems to submit (100 points)

Show your work and explain your reasoning. A correct answer without adequate justification may receive little or no credit. Write in complete, grammatically sound sentences and submit legible work. The number in square brackets is the number of points each problem is worth.

(1) **[15]** DMC Problem 19.11
(2) **[5]** DMC Problem 19.18 [Assume the number of children $n > 0$ is a fixed constant.]
(3) **[10]** DMC Problem 19.19
(4) **[20]** DMC Problem 19.38 (a), (b) [10 points each.]
(5) **[10]** DMC Problem 20.20 (b)
(6) **[20]** DMC Problem 20.25 (c)
(7) **[10]** DMC Problem 20.29 (b)
(8) **[10]** DMC Problem 20.55 [The problem asks for you to define random variables X, Y where X and Y are not independent, but $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$. Hint: You may want to try taking $X \in \{-1, 0, 1\}$ and $Y \in \{0, 1\}$.]

Bonus: Records and cycles in random permutations

This problem is completely optional and worth up to **10 bonus points**, for a maximum homework score of **110/100**. It reveals a surprising fact: two very different features of a random permutation have *exactly the same distribution*.

Let $\pi = (\pi_1, \ldots, \pi_n)$ be a uniformly random permutation of $\{1, \ldots, n\}$. Position i is a *record* if π_i is larger than every preceding entry; position 1 is always a record. For example, $(3, 1, 5, 2, 7, 4, 6)$ has records at positions $1, 3, 5$. Define $R_i = 1$ if position i is a record and $R_i = 0$ otherwise, and let $X_n = R_1 + \cdots + R_n$ be the number of records.

(a) **[1 bonus point]** Show that $\Pr[R_i = 1] = 1/i$.

(b) **[2 bonus points]** Prove that $R_1, \ldots, R_n$ are mutually independent.

[Let r_i be the rank of π_i among $\pi_1, \ldots, \pi_i$, with rank 1 smallest. Show that $\pi \mapsto (r_1, \ldots, r_n)$ is a bijection onto the integer sequences satisfying $1 \le r_i \le i$.]

(c) **[1 bonus point]** Define the *generating polynomial* of the number of records by

$$F_n(z) = \mathbb{E}[z^{X_n}] = \sum_{k=1}^{n} \Pr[X_n = k] z^k.$$

Use part (b) to prove

$$F_n(z) = \frac{z(z+1)\cdots(z+n-1)}{n!}.$$

[If R takes values 0 and 1 with $\Pr[R = 1] = p$, then $\mathbb{E}[z^R] = (1-p) + pz$.]

(d) **[1 bonus point]** View a permutation as the function $\pi(i) = \pi_i$. Write $\pi^t(i)$ for the result of applying π exactly t times; for example, $\pi^2(i) = \pi(\pi(i))$. Because π is a permutation of a finite set, there is a smallest positive integer ℓ for which $\pi^\ell(i) = i$. The sequence $i, \pi(i), \ldots, \pi^{\ell-1}(i)$ is a *cycle of length* ℓ. Every permutation decomposes into a *unique collection of disjoint cycles*, up to the order of the cycles and the starting point of each. For example, $(2, 1, 4, 5, 3, 6)$ has cycle decomposition $(1\ 2)(3\ 4\ 5)(6)$, including the length-one cycle (6). Write the cycle decompositions of $(3, 1, 2, 5, 4)$ and $(2, 3, 4, 5, 1)$.

(e) **[2 bonus points]** Let $a(n, k)$ be the number of permutations of n elements having exactly k cycles. Separate permutations according to whether n forms a cycle by itself. In the other case, explain how to insert n into a cycle of a permutation of $n-1$ elements, and why there are exactly $n-1$ insertion positions. Derive a recurrence expressing $a(n, k)$ in terms of $a(n-1, k-1)$ and $a(n-1, k)$.

(f) **[2 bonus points]** Let Y_n be the number of cycles in a uniformly random permutation, and define

$$G_n(z) = \mathbb{E}[z^{Y_n}] = \sum_{k=1}^{n} \Pr[Y_n = k] z^k.$$

Use part (e) to prove that $G_n(z) = F_n(z)$. Conclude that X_n and Y_n have the same probability distribution, even though they count different features.

(g) **[1 bonus point]** Find the probability that a uniformly random permutation has exactly two cycles. Express your answer using $H_{n-1} = \sum_{j=1}^{n-1} 1/j$.