Use the PDF problem set posted to the course website; its problem numbering differs from the textbook.

Recitation problems (ungraded)

Attempt these before recitation. We will use recitation to work through selected solutions and the ideas behind them; they are chosen to prepare you for the graded problems below.

(1) DMC Problem 1.4 (a), (b)
(2) DMC Problem 1.9
(3) DMC Problem 2.2
(4) DMC Problem 2.20 (a), (b)
(5) DMC Problem 2.22 (a), (c)
(6) DMC Problem 3.4
(7) DMC Problem 3.25 (a)–(c)
(8) DMC Problem 3.23 (a)
(9) DMC Problem 3.38 (b), (d), (e)
(10) DMC Problem 3.43 (c), (d)

Problems to submit (100 points)

Show your work and explain your reasoning. A correct answer without adequate justification may receive little or no credit. Write in complete, grammatically sound sentences and submit legible work. The number in square brackets is the number of points each problem is worth.

(1) **[8]** DMC Problem 1.4 (c), (d)
(2) **[10]** DMC Problem 1.10
(3) **[8]** DMC Problem 2.3 (c), (d)
(4) **[8]** DMC Problem 2.20 (c), (d)
(5) **[10]** DMC Problem 2.24
(6) **[8]** DMC Problem 3.5 (d), (f), (h)
(7) **[10]** DMC Problem 3.26 (a)–(d), (h)
(8) **[10]** DMC Problem 3.31 (a), (b)
(9) **[12]** DMC Problem 3.40
(10) **[16]** DMC Problem 3.41

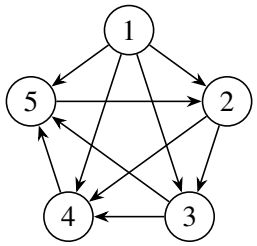
Bonus challenge: Tournaments (up to 10 bonus points)

This problem is completely optional and is meant to be substantially more challenging than the required homework. Each part is graded all-or-nothing: a complete proof earns the listed bonus points, while an incomplete proof earns no points for that part. Bonus points are added to your homework score, up to a maximum score of **110/100**.

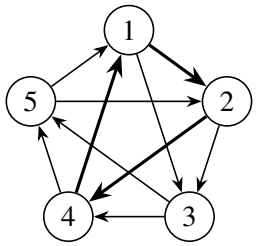
A *tournament* is a directed graph that records the outcome of every possible pairwise comparison. Tournaments arise naturally in round-robin competitions, but the same mathematical model appears in computer science whenever objects are compared pairwise—for example, in ranking systems, preference aggregation, and algorithms that must reason about possibly inconsistent comparisons. A basic question is: what structure must appear when the pairwise comparisons do not produce a single object that beats everything else?

Formally, let the vertices be $[n] = \{1, 2, \ldots, n\}$. A tournament T contains exactly one directed edge between every pair of distinct vertices: for each $1 \le i < j \le n$, exactly one of (i, j) and (j, i) belongs to T. If $(i, j) \in T$, we say that i *beats* j.

The three-vertex cycle is the smallest example with no winner, but larger tournaments already show much richer behavior. The two examples below use five vertices. Every pair of vertices is joined by exactly one directed edge, so each picture is a tournament.



A tournament with a winner. Vertex 1 beats all four other vertices, so 1 is a winner. Notice that the comparisons among $2, 3, 4, 5$ can still contain cycles; having a winner does not mean that all pairwise comparisons are globally consistent.



A tournament with no winner. Each vertex beats exactly two others, so no vertex is a winner. The heavy edges form the directed cycle $(1, 2, 4)$. This cycle is *uncovered*: the two vertices outside it are both beaten by a vertex on the cycle—1 beats 3, and 4 beats 5.

A *winner* is a vertex i that beats every other vertex $j \ne i$. A *cycle of length* L is a sequence of distinct vertices $(i_1, i_2, \ldots, i_L)$ such that

$$i_1 \text{ beats } i_2, \quad i_2 \text{ beats } i_3, \quad \ldots, \quad i_{L-1} \text{ beats } i_L, \quad i_L \text{ beats } i_1.$$

(a) **[2 bonus points]** Prove that if a tournament has no winner, then it contains a cycle.

(b) **[2 bonus points]** Strengthen part (a): prove that if a tournament has no winner, then it contains a cycle of length 3.

(c) **[5 bonus points]** Call a cycle $(i_1, i_2, \ldots, i_L)$ *uncovered* if every vertex outside the cycle loses to at least one vertex on the cycle. In other words, for every

$$j \in [n] \setminus \{i_1, i_2, \ldots, i_L\},$$

there is some $k \in \{1, \ldots, L\}$ such that i_k beats j. Prove that if a tournament has no winner, then it contains an uncovered cycle.

(d) **[1 bonus points]** Suppose $n \le 5$ and the tournament has no winner. Prove that it contains an uncovered cycle of length 3.

For perspective: when n is large, there are tournaments with no winner in which every uncovered cycle has length at least 4. Finding such an example is itself a nontrivial challenge.