QUIZ 2: 110 Minutes

Last Name: Sample Solution

First Name: _____

RIN: _____

Section: _____

Write your name and RIN clearly above and on your crib sheet.

Answer **ALL** questions and show enough work for us to follow your reasoning, especially on proofs.

You have until 10:00am to submit your exam, unless your accommodations include extended time.

NO COLLABORATION or electronic devices, including calculators, are permitted. Violations will result in a grade of **F**.

You may ask for clarification about wording or notation. We cannot provide hints or confirm whether an approach is correct.

You may use the back of exam pages for scratch work if you fill up the attached scratch sheets.

When finished, bring your exam, crib sheet, and ID to the front. Please wait for the proctor to confirm that you are all set before leaving.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	4	Total
50	60	60	40	210

1 True or False [50 points]

Circle one answer per question. **5 points** for each correct answer.
(You do not need to justify your answers, but include any scratch work.)

(a) Let p be a prime and a, b, c be integers, with $a \not\equiv 0 \pmod{p}$. If $ab \equiv ac \pmod{p}$, then $b \equiv c \pmod{p}$.

(True) False

$ab \equiv ac \pmod{p} \iff p \mid a(b-c)$
and since p is prime, $\gcd(p, a) = 1$
which means $p \mid b-c$
$\iff b \equiv c \pmod{p}$

(b) Let p be a prime and a, b be integers, with $a, b \not\equiv 0 \pmod{p}$. If $a^2 \equiv b^2 \pmod{p}$, then $a \equiv b \pmod{p}$.

True (False)

not true, let $p=3$, $a=1$, $b=2$
then $a^2 \equiv 1 \pmod{p}$ and
$b^2 = 4 \equiv 1 \pmod{p}$ so
$a^2 \equiv b^2 \pmod{p}$ but $a \not\equiv b \pmod{p}$

(c) We have $(2n)^2 \in O(n^2)$. [Recall: $f(n) \in O(g(n))$ means there exist constants $C, n_0 > 0$ such that $f(n) \le Cg(n)$ whenever $n \ge n_0$.]

(True) False

$\frac{(2n)^2}{n^2} = 4$ so $(2n)^2 \le 4 \cdot n^2$
for $n \ge 1$
which means $(2n)^2 \in O(n^2)$

(d) We have $2^{2n} \in O(2^n)$.

True (False)

$\frac{2^{2n}}{2^n} = 2^n \to \infty$ as $n \to \infty$ so
$2^{2n} \in \omega(2^n)$

1 True or False (continued)

(e) We have $\sqrt{n}\log_2 n \in O(n)$.

[True] [False]

$$\frac{\sqrt{n}\log_2 n}{n} = \frac{\log_2 n}{\sqrt{n}} < \frac{n^{1/4}}{n^{1/2}} = \frac{1}{n^{1/2}} \downarrow 0$$

when n is sufficiently large that $\log_2 n < n^{1/4}$, which is true when $n > 2^{16}$

(f) Every graph with 6 vertices and 5 edges is a tree.

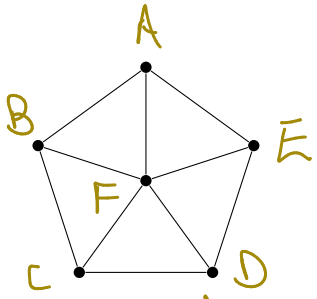
[True] [False]

counterexample:



(g) The graph on the right can be 3-colored.

[True] [False]



No: assume there were a valid three coloring, then WLOG we can assume it colors F with 1, and A with 2. Then B, E must be colored 3 to not have the same color as A, and C and D must be colored 2 to have different colors than B and E. But since C and D are neighbors, this shows this supposedly valid coloring is not valid. This contradiction implies $\chi(G) > 3$.

1 True or False (continued)

(h) The number of different strings formed by rearranging all the letters of BANANA is $\frac{6!}{3!\,2!}$.

[True] [False]

There are: 1 letter of type B
3 letters of type A
2 letters of type N

$\Rightarrow \binom{6}{1,2,3} = \frac{6!}{3!\,2!\,1!}$ ways to rearrange letters

(i) The number of different ways to place n identical balls into m distinct bins is $\frac{n^m}{m!}$.

[True] [False]

Let X_i denote the number of balls in bin i
Then we're asking how many ways to find nonnegative integral solutions to
$x_1 + \cdots + x_m = n$, i.e. $\binom{n+m-1}{m-1} \neq \frac{n^m}{m!}$

(j) If $A \cap B \cap C = \emptyset$, then $|A \cup B \cup C| = |A| + |B| + |C|$.

[True] [False]

Use PIE:

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

and $|A \cap B \cap C| = |\emptyset| = 0$, so

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C|$$

2 Multiple choice [60 points]

Circle **one answer** per question. **15 points** for each correct answer.
Show your work for each problem: a correct answer with no work may earn zero points.

(a) Let P be the set of binary strings generated by the rules on the right (where we recall that $\bullet$ denotes string concatenation). Which of the following are ***not*** in P?

① $0 \in P$,
$1 \in P$.

② $x \in P \implies x \bullet 0 \bullet x \in P$,
$x \in P \implies x \bullet 1 \bullet x \in P$.

[A] 1110111

[B] 0100010

[C] 1010101

[D] 1001001

[E] All of the above strings are in P.

A: $1 \in P \underset{\text{rule 2}}{\Rightarrow} 111 \in P \underset{\text{rule 2}}{\Rightarrow} 1110111 \in P$

B: $1 \in P \underset{\text{rule 2}}{\Rightarrow} 010 \in P \underset{\text{rule 2}}{\Rightarrow} 0100010 \in P$

C: $0 \in P \underset{2}{\Rightarrow} 101 \in P \underset{2}{\Rightarrow} 1010101 \in P$

D: $100 \notin P \Rightarrow 1001001 \notin P$

(b) Which of the following is true of the function $f(n) = 1 + 2 + 2^2 + \ldots + 2^n = \sum_{i=0}^{n} 2^i$?

[A] $f \in O(n^2)$.

[B] $f \in O(2^n)$, but $f \notin O(n^2)$.

[C] $f \in O(2^{n+1})$, but $f \notin O(2^n)$.

[D] $f \in O(n \cdot 2^{n+1})$, but $f \notin O(2^{n+1})$.

by common sum, $f(n) = 2^{n+1} - 1$

this grows faster than n^2

so $f(n) \notin O(n^2)$

also,

$\frac{f(n)}{2^n} = 2 - \frac{1}{2^n} < 2$ when $n \geq 1$

so $f(n) \in O(2^n)$

2 Multiple choice (continued)

(c) When is it true that $n^3 \equiv n \pmod 6$?

A If $n \equiv 2 \pmod 6$, then $n^3 \equiv n \pmod 6$.

B If $n \equiv 3 \pmod 6$, then $n^3 \equiv n \pmod 6$.

C If $n \equiv 4 \pmod 6$, then $n^3 \equiv n \pmod 6$.

D If $n \equiv 5 \pmod 6$, then $n^3 \equiv n \pmod 6$.

E All of the above are true.

F None of the above are true.

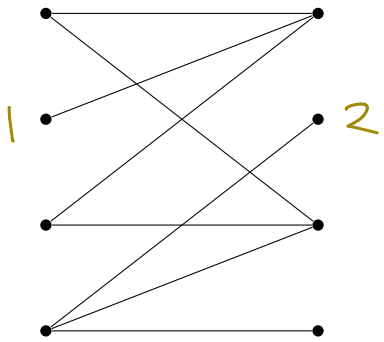
$n \equiv 2 \pmod 6 \Rightarrow n^3 \equiv 8 \pmod 6 \Rightarrow n^3 \equiv 2 \pmod 6$

$n \equiv 3 \pmod 6 \Rightarrow n^3 = 27 \pmod 6 \Rightarrow n^3 = 3 \pmod 6$

$n \equiv 4 \pmod 6 \Rightarrow n^3 = 64 \pmod 6 \Rightarrow n^3 = 4 \pmod 6$

$n \equiv 5 \pmod 6 \Rightarrow n^3 = 125 \pmod 6 \Rightarrow n^3 = 5 \pmod 6$

(d) Let G be the graph on the right. Which of the following are true?



A G is planar.

B G has an odd-length cycle.

C G has a matching of size 4.

D G is a tree.

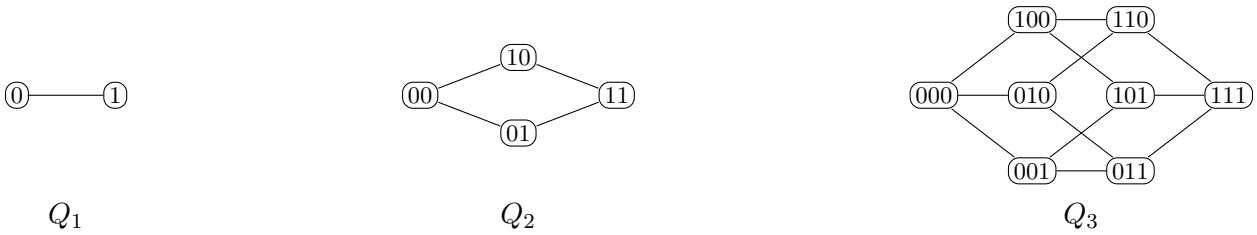
E None of the above are true.

G is planar: move nodes 1 and 2 placement to avoid overlapping edges

3

[60 points]

For integer $n \geq 1$, define a graph Q_n as follows. The vertices of Q_n are all 2^n binary strings of length n. Vertices s and t are joined by an edge in Q_n if and only if strings s and t differ in exactly one position. For example, 0010 and 1010 are adjacent in Q_4, while 0010 and 1011 are not adjacent in Q_4.



Q_1 Q_2 Q_3

(a) **[30 points]** Prove that for every $n \geq 1$, the graph Q_n is bipartite.
[Recall: A graph is bipartite if there are subsets L, R of vertices with $L \cap R = \emptyset$ such that every edge is incident to one vertex in L and one vertex in R.]

Let L be the set of strings with an even number of 1s and R be those strings with an odd number of 1s.

Since (u,v) is an edge only if one bit is flipped between u and v, one of the strings has an even number of 1s and the other has an odd number of 1s for every edge in Q_n.

This shows Q_n is bipartite as all edges have one vertex in L and the other in R

3 **(continued)**

(b) **[30 points]** How many edges does Q_n have? Justify your answer.

Every vertex in Q_n has degree n (b/c each neighbor corresponds to flipping a single bit) and there are 2^n vertices, so by the handshaking lemma,

$$\sum_{v \in Q_n} n = 2^n \cdot n = 2|E|$$

so

$$|E| = 2^{n-1} \cdot n$$

4

[40 points]

Suppose you have $n \geq 5$ distinct jewelry beads, and you want to make a bracelet with 5 of them. You pick the 5 beads, arrange them in some order on a string, and join the string in a loop such that the beads can slide around anywhere on the loop. The resulting bracelet is the same even if we rotate the loop, or flip the loop over. For some examples (with $n = 8$), the first three bracelets are all identical, but the fourth one is different.



How many distinct possible bracelets are there? Justify your answer.

We construct the bracelet by

(i) Choosing 5 distinct beads, N_1 ways

(ii) Ordering the 5 beads, N_2 ways

So there are $N = N_1 \times N_2$ distinct bracelets.

Here $N_1 = \binom{n}{5}$.

To evaluate N_2 note that there are $5!$ ways to arrange the beads in a row, and there are 5 ways to order that row of beads on a string so a distinct bead is the "first" on the bracelet and further there are two reflections around that "first" bead, so by the multiplicity rule,

$$N_2 = \frac{5!}{10} \Rightarrow N = \binom{n}{5} \cdot \frac{5!}{10}$$

SCRATCH

SCRATCH