# Quiz 2 — Answer Sheet

CSCI 2200: Foundations of Computer Science. Fall 2026. RPI. 210 points total.

# 1. True or false

**Key:** (a) True (b) False (c) True (d) False (e) True
(f) False (g) False (h) True (i) False (j) False

**Brief reasoning:**

(a) **True.** Since $p$ is prime and $a \not\equiv 0 \pmod{p}$, we know $a$ has a multiplicative inverse modulo $p$. Multiplying $ab \equiv ac \pmod{p}$ by this inverse gives $b \equiv c \pmod{p}$. Another way to see this is the following. We know $ab - ac = a(b - c)$ is divisible by $p$, but $a$ is not divisible by $p$. Hence, Euclid's lemma says that $b - c$ is divisible by $p$.

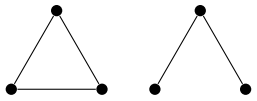
(b) **False.** We can see why this is false as follows. If $a^2 \equiv b^2 \pmod{p}$, then $a^2 - b^2 = (a - b)(a + b)$ is divisible by $p$. Hence, Euclid's lemma says that $a - b$ *or* $a + b$ is divisible by $p$. In the second case, $a \equiv -b \pmod{p}$, but we need not have $a \equiv b \pmod{p}$. For a concrete example, let $a = 2$, and $b = 3$, and $p = a + b = 5$. We have $a^2 = 4$ and $b^2 = 9 \equiv 4 \equiv a^2 \pmod{5}$, but $a \not\equiv b \pmod{5}$.

(c) **True.** We have $(2n)^2 = 4n^2$. In the definition of $O(n^2)$, take $C = 4$ and $n_0 = 1$.

(d) **False.** The ratio $2^{2n}/2^n = 2^n$ is unbounded. Thus no constant $C$ can satisfy $2^{2n} \leq C2^n$ for every sufficiently large $n$.

(e) **True.** A logarithm grows more slowly than $\sqrt{n}$. In particular, $\log_2 n \leq \sqrt{n}$ for $n \geq 16$, so $\sqrt{n} \log_2 n \leq n$ for all such $n$.

(f) **False.** Take a triangle together with a separate path of length 2, as shown on the right. This graph has $3 + 3 = 6$ vertices and $3 + 2 = 5$ edges, but it is not a tree: it contains a cycle and is not connected.



(g) **False.** The five vertices on the outer cycle require at least three colors because the cycle has odd length. The center is adjacent to all five, so it needs a fourth color.

(h) **True.** If all the six letters were distinct, there would be 6! possible arrangements. We can permute the three identical A's in 3! ways without changing the string, and permute the two identical N's in 2! ways without changing the string. Hence dividing the 6! labeled arrangements by 3!2! gives $6!/(3!2!) = 60$ distinct strings.

(i) **False.** For example, when $n = 1$ and $m = 2$, there are two placements, whereas the formula results in a fraction: $n^m/m! = 1/2$. The correct general count, when empty bins are allowed, is $\binom{n+m-1}{m-1}$, by a "stars and bars" argument.

(j) **False.** The formula would be true if $A \cap B = \emptyset$ and $A \cap C = \emptyset$ and $B \cap C = \emptyset$. However, let $A = \{1\}$, $B = \{1, 2\}$, and $C = \{2\}$. The sets' intersection $A \cap B \cap C$ is empty. However, $|A \cup B \cup C| = 2$ and $|A| + |B| + |C| = 1 + 2 + 1 = 4$.

# 2. Multiple choice

**Key:** (a) $\boxed{\text{D}}$ (b) $\boxed{\text{B}}$ (c) $\boxed{\text{E}}$ (d) $\boxed{\text{A}}$

**Brief reasoning:**

(a) **D.** Any generated string of length seven must have the form $x \bullet b \bullet x$, where $x$ has length three and $b \in \{0, 1\}$. In 1001001, the first three bits are 100 and the last three are 001, so the two copies of $x$ would differ. Hence this string is not in $P$.

The other strings are all in $P$, as we can see via the following derivations:

$$\begin{aligned} 1 \in P &\implies 111 \in P \implies 1110111 \in P, \\ 0 \in P &\implies 010 \in P \implies 0100010 \in P, \\ 1 \in P &\implies 101 \in P \implies 1010101 \in P. \end{aligned}$$

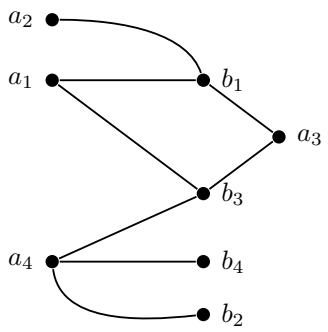
(b) **B.** The geometric-sum formula gives

$$f(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1.$$

Thus $2^n \leq f(n) < 2 \cdot 2^n$, so $f \in \Theta(2^n)$, and in particular $f \in O(2^n)$. Exponentials grow much faster than quadratics, so $f \notin O(n^2)$, so option A is false and option B is true. We also have $f \in O(2^{n+1})$ (and indeed $O(2^{n+1}) = O(2^n)$), but options C and D are false because option B is true.

(c) **E.** One can prove that each of A through D are true via a direct modular arithmetic calculation. We can also prove directly that $n^3 \equiv n \pmod 6$ for all $n$, as follows. For every integer $n$, we have $n^3 - n = n(n-1)(n+1)$. Among three consecutive integers, at least one is even and at least one is divisible by 3. Therefore their product is divisible by 6, as needed.

(d) **A.** Label the left nodes $a_1, a_2, a_3, a_4$ and the right nodes $b_1, b_2, b_3, b_4$, from top to bottom (as in the original drawing). The drawing on the right shows that this graph is planar.




B is false because the graph is bipartite, and hence any cycle must be even-length (indeed, the only cycle is $a_1, b_1, a_3, b_3$). C is false because the top three left vertices, $a_1, a_2, a_3$, have only two possible neighbors on the right, $b_1, b_3$; hence Hall's theorem says we cannot find a matching that covers all 4 left vertices. D is false because the graph has a cycle.

# 3. A graph on binary strings

Let $V$ denote the vertices of $Q_n$, and $E$ denote the edges.

**(a).** Consider any edge $(s, t)$ in $E$. By definition, the strings $s$ and $t$ differ in exactly one position, say position $i$. By swapping the names of the endpoints if necessary, we may assume that $s[i] = 0$ and $t[i] = 1$, while $s[j] = t[j]$ for every index $j \neq i$. Hence, going from $s$ to $t$ increases the total number of "1"s in the string by exactly one.

Motivated by the above, let $L$ denote the set of all length-$n$ strings with an even number of "1"s, and let $R$ denote those with an odd number of "1"s. Every string has either an even or an odd number of 1s, so these sets are disjoint and together contain all the vertices.

Since each edge $(s, t)$ is such that $t$ has exactly one more 1 than $s$ (or vice-versa), the two strings cannot both have an even number of 1s, nor can they both have an odd number. If $s$ is in $L$, then $t$ is in $R$; if $s$ is in $R$, then $t$ is in $L$. Hence, the edge crosses from one side to the other.

This is true for every edge, so $Q_n$ is bipartite. □

While the above is (likely) the cleanest way to phrase this proof, there are some reasonable alternatives, such as considering cycles[1] or doing induction[2].

[1]Here is a terse version of an argument similar to the above, but using cycles. We know that a graph is bipartite if and only if it contains no odd-length cycles. Consider any cycle in $Q_n$. To return to the starting string, each coordinate must be flipped an even number of times. Since each edge flips exactly one coordinate, the cycle's length is a sum of even numbers and is even. Hence $Q_n$ has no odd cycle and is bipartite.

[2]Here is a terse version of a proof by induction. The graph $Q_1$ is a single edge, so it is bipartite. Suppose $Q_n$ has bipartition $(L, R)$. We claim that

$$L' = \{0x : x \in L\} \cup \{1x : x \in R\}, \qquad R' = \{0x : x \in R\} \cup \{1x : x \in L\}$$

is a bipartition of $Q_{n+1}$. Proof: Any edge in $Q_{n+1}$ is of the form $(0x, 1x)$, or $(bx, by)$ where $b \in \{0, 1\}$ and $(x, y)$ is an edge in $Q_n$. In the first case, the edge $(0x, 1x)$ crosses $L', R'$ by definition; in the second case, the edge $(bx, by)$ crosses $L', R'$ by the inductive hypothesis. This completes the proof by induction.

**(b).** Consider any vertex $s = s[1]s[2]\cdots s[n]$. It has one neighbor obtained by flipping the first bit $s[1]$, another obtained by flipping the second bit $s[2]$, and so on, up to the neighbor obtained by flipping $s[n]$. Hence, the degree of every vertex $v$ is $\deg(v) = n$.

Moreover, there are $2^n$ vertices. Recall the "handshaking lemma": if we add up the degrees of all vertices, every edge is counted twice (once for each of its two endpoints). Hence, half of the sum of the degrees equals the number of edges:

$$|E| = \frac{1}{2}\sum_{v\in V}\deg(v) = \frac{1}{2}\sum_{v\in V} n = \frac{1}{2}\cdot 2^n \cdot n = \boxed{n2^{n-1}}. \qquad \square$$

While the above is (likely) the cleanest way to phrase this proof, there are some reasonable alternatives, such as directly counting the edges[3] or by solving a recurrence[4].

[3]Here is a terse proof using a direct counting argument. We count edges $(s,t)$ by choosing (1) one of the $n$ possible indices in which the strings $s$ and $t$ will differ, then choosing (2) the bits in the remaining $n-1$ that will be the same in both $s$ and $t$. There are $n$ ways to make the choice in (1) of an index $i$ where, say, $s[i] = 0$ and $t[i] = 1$. Then there are $2^{n-1}$ ways to choose the remaining bits in (2), for a total count of $n \cdot 2^{n-1}$.

[4]Here is a proof that solves a recurrence using induction. Let $e_n$ denote the number of edges in $Q_n$. We have $e_1 = 1$. For $n \geq 2$, observe that $Q_n$ contains two copies of $Q_{n-1}$: one with all vertices of the form $0x$ and one with all vertices of the form $1x$. These copies of $Q_{n-1}$ each have $e_{n-1}$ "internal" edges, i.e., edges where both vertices are in the same copy. There is also one edge $0x$–$1x$ for each length-$(n-1)$ string $x$, giving $2^{n-1}$ additional edges. These cases cover all edges of $Q_n$; hence $e_n = 2e_{n-1} + 2^{n-1}$.

We now verify $e_n = n2^{n-1}$ by induction. The formula holds for the base case of $n = 1$. Assuming it holds at $n-1$, the recurrence gives

$$e_n = 2\big((n-1)2^{n-2}\big) + 2^{n-1} = (n-1)2^{n-1} + 2^{n-1} = n2^{n-1}.$$

This completes the induction.

# 4. Five-bead bracelets

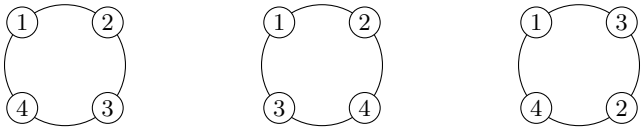
There are $S = n(n-1)(n-2)(n-3)(n-4) = n!/(n-5)!$ ordered sequences of five distinct beads from among the $n$ possible beads.[5] This is the same as the number of ways to arrange 5 beads on a string from left to right (with a distinguished left end of the string), before joining the string in a loop to form the bracelet.

Now, let $B$ denote the total number of distinct bracelets. For each bracelet, there are exactly 10 corresponding sequences of beads on the string: 5 for each way to rotate the bracelet and 5 for each way to rotate the bracelet *after being flipped*.[6] Hence we have $10 \cdot B = S$, and thus

$$B = \boxed{\frac{n(n-1)(n-2)(n-3)(n-4)}{10}}$$

distinct bracelets.[7] □

For a smaller example of how the above counting works, suppose we had $n = 4$ beads and used all of them to make a bracelet. There are $4! = 24$ orderings of the beads, but for each bracelet there are $4 \cdot 2$ corresponding orderings. Hence there are 3 distinct bracelets, specifically:




[5] Another reasonable way to denote this quantity is $S = \binom{n}{5} \cdot 5!$, which corresponds to first choosing 5 beads from among the $n$ beads, then ordering them in one of the 5! possible ways.

[6] In other words, when we count sequences, each bracelet appears 10 times: corresponding to choosing 5 possible starting positions of the sequence, and 2 possible "directions" (flipped or not flipped).

To be very concrete: Let $b = (b_0, b_1, b_2, b_3, b_4)$ denote some sequence of beads. Then, for each $i \in \{0, 1, 2, 3, 4\}$, the sequence $(b_i, b_{i+1}, b_{i+2}, b_{i+3}, b_{i+4})$, where indices are taken mod 5, corresponds to the same bracelet as $b$. Additionally, for each $i \in \{0, 1, 2, 3, 4\}$, the sequence $(b_i, b_{i-1}, b_{i-2}, b_{i-3}, b_{i-4})$, where indices are taken mod 5, corresponds to the same bracelet as $b$. These are all sequences corresponding to the same bracelet as $b$.

[7] If we continued to phrase things in terms of $\binom{n}{5}$, our final answer is $B = \frac{5!}{10} \cdot \binom{n}{5} = 12 \cdot \binom{n}{5}$.

One reasonable alternative argument that also arrives at this quantity goes as follows. We can enumerate bracelets by first choosing 5 unordered beads in one of $\binom{n}{5}$ possible ways. Now, let us rotate the bracelet so that the smallest-index bead is "on top," and consider only the position of the other beads. There are 4! orderings for the remaining beads, but 2 such orderings correspond to each distinct bracelet (one for each way to reverse the sequence; corresponding to flipping the bracelet over). Hence there are $4!/2 = 12$ bracelets for each chosen set, giving $12\binom{n}{5}$ in total.