Quiz 1 — Answer Sheet

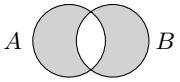
CSCI 2200: Foundations of Computer Science. Fall 2026. RPI. 330 points total.

1. Multiple choice (12 points each)

Key: (a) C (b) B (c) C (d) A (e) A (f) A (g) A (h) B (i) B (j) B

Brief Reasoning:

(a) **C.** The given set consists of points either (1) in A but not B, or (2) in B but not A. Equivalently, these are points in $A \cup B$ but not $A \cap B$.



(b) **B.** For example, 1 is in both sets: take $k = 1/2$ and $j = 0$. In contrast, $\sqrt{5}$ is irrational, and an integer square cannot be prime.

(c) **C.** The implication is `F` exactly when r is `F` and $p \vee q$ is `T`. There are three such assignments (those where $r =$ `F` and one or both of p and q are `T`) out of eight. In the other five rows, the implication is `T`.

(d) **A.** Since one of p or $\neg p$ is always `F`, we know $p \wedge \neg p$ is always `F`; the same will be true of $p \wedge q \wedge \neg p$. The other options can be true: for B take p false; C is always true; for D take p false.

(e) **A.** We have $P(0), P(1), P(2)$ and, combining the given rule for even n with A's rule for odd n, $P(n) \Rightarrow P(n+3)$ for every $n \geq 0$. B and C do not establish $P(2)$; for example, under either B or C, $P(n)$ may only hold at $n = 0, 1, 3$. D establishes only that $P(n)$ holds for n a multiple of 3.

(f) **A.** If L means late, Q means a quiz occurs, and E means it is easy, the contrapositive of $L \to (Q \wedge E)$ is $\neg(Q \wedge E) \to \neg L$, which is equivalent to $(\neg Q \vee \neg E) \to \neg L$.

(g) **A.** Since $n^2 + n = n(n+1)$, an even factor makes the product even; this gives a direct proof. B rewrites the expression in a way that could facilitate a proof by induction; C attempts to start a proof by contradiction but asserts an invalid algebraic consequence of $n^2 + n = 2k + 1$.

(h) **B.** We use strong induction; the base case $n = 1$ holds by definition. If $n > 1$ is odd, use $k = 0$ and $\ell = n$. If $n > 1$ is even, apply the inductive hypothesis to $n/2 < n$; multiplying its representation by 2 gives one for n. This works because $1 \leq n/2 < n$, while the approach in A does not apply to $n/2$; C excludes $m = n/2$; and D excludes the case where $m = n/2$ is even.

(i) **B.** The negation of $\exists i : \ x_i \geq \overline{x}$ is $\forall i : \ x_i < \overline{x}$. (As an aside: We can complete the proof by contradiction as follows. Summing the strict inequalities gives $\sum_i x_i < n\overline{x} = \sum_i x_i$, a contradiction.)

(j) **B.** If $n \geq 2$ is composite, $n = ab$ with $2 \leq a, b < n$. Strong induction supplies prime factorizations for a and b; concatenate them.

2. An inequality (100 points)

(a) [30 points]. For $n \geq 1$, we have

$$\frac{1}{n} - \frac{1}{n+1} = \frac{n+1}{n(n+1)} - \frac{n}{n(n+1)} = \frac{1}{n(n+1)} \geq \frac{1}{(n+1)^2},$$

since $n(n+1) \leq (n+1)^2$.

(b) [70 points]. Let $S_n = \sum_{i=1}^{n} 1/i^2$. We prove $S_n \leq 2 - 1/n$ by induction on n.

For the base case, let $n = 1$. Then $S_1 = 1 \leq 1 = 2 - 1$. This prove the base case.

Now assume by induction that $S_n \leq 2 - 1/n$. We have:

$$\begin{aligned} S_{n+1} &= S_n + \frac{1}{(n+1)^2} && \text{(by definition)} \\ &\leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} && \text{(by the inductive hypothesis)} \\ &\leq 2 - \frac{1}{n} + \left(\frac{1}{n} - \frac{1}{n+1}\right) && \text{(by part (a))} \\ &= 2 - \frac{1}{n+1}. \end{aligned}$$

Hence, if the statement holds for n, it also holds for $n+1$. Thus, by induction the statement holds for every integer $n \geq 1$.

3. An integer sum-of-powers (110 points)

Fix x and assume that $x + x^{-1} \in \mathbb{Z}$.

For our base cases, consider $n = 0$ and $n = 1$. When $n = 0$, $x^n + x^{-n}$ is simply $x^0 + x^0 = 1 + 1 = 2 \in \mathbb{Z}$. When $n = 1$, we know $x^1 + x^{-1}$ is an integer by assumption. [Note that *two* base cases are needed for the proof to be valid.[1]]

Now, consider some $n \geq 1$, and assume by strong induction that the result holds for all $m \leq n$; i.e., assume that for all $0 \leq m \leq n$, we have $x^m + x^{-m} \in \mathbb{Z}$. Multiplication and cancellation yield

$$(x^n + x^{-n})(x + x^{-1}) = x^{n+1} + x^{n-1} + x^{-n+1} + x^{-n-1} = \left(x^{n+1} + x^{-(n+1)}\right) + \left(x^{n-1} + x^{-(n-1)}\right)$$

By assumption, $x + x^{-1}$ is an integer, and under the inductive hypothesis, both $x^n + x^{-n}$ and $x^{n-1} + x^{-(n-1)}$ are integers. Hence,

$$x^{n+1} + x^{-(n+1)} = (x^n + x^{-n})(x + x^{-1}) - \left(x^{n-1} + x^{-(n-1)}\right)$$

is an integer.

Together with the two base cases, this completes the proof by strong induction.

[1] In more detail, this is because our proof shows that for all $n \geq 1$, we have $P(n-1) \wedge P(n) \implies P(n+1)$. Hence multiple base cases are needed.