QUIZ 1: 110 Minutes

Last Name: _____

First Name: _____

RIN: _____

Section: _____

Write your name and RIN clearly above and on your crib sheet.

Answer **ALL** questions and show enough work for us to follow your reasoning, especially on proofs.

You have until 10:00am to submit your exam, unless your accommodations include extended time.

NO COLLABORATION or electronic devices, including calculators, are permitted. Violations will result in a grade of **F**.

You may ask for clarification about wording or notation. We cannot provide hints or confirm whether an approach is correct.

You may use the back of exam pages for scratch work if you fill up the attached scratch sheets.

When finished, bring your exam, crib sheet, and ID to the front. Please wait for the proctor to confirm that you are all set before leaving.

GOOD LUCK!

You **MUST** show **CORRECT** work to get full credit.

When in doubt, TINKER.

1	2	3	Total
120	100	110	330

1 **Circle one answer per question. 12 points for each correct answer. Show your work for each problem: a correct answer with no work may earn zero points.**

(a) Which of the following is equal, for all sets A and B, to $(A \cap \overline{B}) \cup (B \cap \overline{A})$?

A $(A \cap B) \cup (\overline{A} \cap \overline{B})$

B $\big(A \cap (B \cup \overline{A})\big) \cup \big(B \cap (A \cup \overline{B})\big)$

C $(A \cup B) \cap \big(\ \overline{A \cap B}\ \big)$

D $(A \cup \overline{B}) \cap B$

(b) Which of the following sets is non-empty?

A $\{x \mid x \in \mathbb{Q}, x^2 = 5\}$

B $\{2k \mid k \in \mathbb{Q}\} \cap \{2j + 1 \mid j \in \mathbb{Q}\}$

C $\{n^2 \mid n \in \mathbb{Z}\} \cap \{p \mid p \in \mathbb{N}, p \text{ is prime}\}$

D None of the above.

(c) Consider the truth table for the compound proposition $P = (p \vee q) \to r$. Out of the 8 rows of this truth table, in how many does P have value `T`?

A 3

B 4

C 5

D 6

E None of the above.

(d) A compound proposition is called a *contradiction* if it evaluates to `F` for *every* possible assignment of values (`T` or `F`) to the variables of the proposition. Which of the following is a contradiction?

A $p \wedge (q \wedge \neg p)$

B $p \rightarrow (p \wedge q)$

C $p \vee \neg p$

D $(p \wedge q) \rightarrow (r \vee s)$

E None of the above.

(e) Suppose we want to prove that $P(n)$ holds for all integers $n \geq 0$. We already know that $P(0)$ is true and that

$$P(n) \implies P(n+3)$$

for every **even** integer $n \geq 0$. Which additional information is sufficient to conclude that $P(n)$ holds for every $n \geq 0$?

A $P(1)$ and $P(2)$, together with $P(n) \implies P(n+3)$ for every **odd** $n \geq 1$.

B $P(1)$, together with $P(n) \implies P(n-2)$ for every $n \geq 2$.

C $P(1)$, together with $P(n) \implies P(n/2)$ for every **even** $n \geq 2$.

D $P(n) \implies P(n+3)$ for every **odd** $n \geq 1$, together with $P(n) \implies P(n/2)$ for every **even** $n \geq 2$.

(f) What is the contrapositive of the statement "If I am late to class, then there will be a pop quiz and it will be easy"?

A If there is no pop quiz or the pop quiz is not easy, then I am not late to class.

B If there is no pop quiz, then I am not late to class.

C If I am not late to class, then there will be no pop quiz or the pop quiz will not be easy.

D If there is a pop quiz and it is not easy, then I am late to class.

E None of the above.

(g) Which of the following observations immediately gives a direct proof that n^2+n is even for every natural number n?

A $n^2 + n = n(n + 1)$, and one of two consecutive integers is even.

B $n^2 + n = (n + 1)^2 - (n + 1)$.

C if $n^2 + n = 2k + 1$ is odd, then $n = k + 1/2$ is not an integer.

D None of the above.

(h) To prove that every integer $n \geq 1$ can be written as

$$n = 2^k \ell,$$

where $k \geq 0$ is an integer and ℓ is an odd integer, suppose the proof handles odd n directly and, when n is even, reduces the problem to $n/2$. Which induction hypothesis directly guarantees that the claim is available for $n/2$?

A Assume the claim holds for $n - 1$.

B Assume the claim holds for every integer m with $1 \leq m < n$.

C Assume the claim holds for every integer m with $n/2 < m < n$.

D Assume the claim holds for every odd integer $m < n$.

E No induction hypothesis is needed in the even case.

(i) Let $\overline{x}$ be the average of real numbers $x_1, \ldots, x_n$. To prove by contradiction that $x_i \geq \overline{x}$ for some $i \in \{1, 2, \ldots, n\}$, which assumption should we make?

A $x_i \leq \overline{x}$ for every i.

B $x_i < \overline{x}$ for every i.

C $x_i > \overline{x}$ for every i.

D $x_i \geq \overline{x}$ for at least one i.

E None of the above.

(j) To prove that every integer $n \geq 2$ is a product of primes, suppose n is not itself prime. Which fact is the useful next step?

A $n = p + q$ for two primes $p, q < n$.

B $n = ab$ for some integers $2 \leq a, b < n$.

C $n = m + 1$ for some integer $m < n$.

D None of the above.

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(a) **[30 points]** Give a direct proof that, for every integer $n \geq 1$,

$$\frac{1}{(n+1)^2} \leq \frac{1}{n} - \frac{1}{n+1}.$$

(b) **[70 points]** Using part (a), prove by induction on n that for all $n \geq 1$,

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}.$$

3

[110 points] Let $x \neq 0$ satisfy $x + \frac{1}{x} \in \mathbb{Z}$. Prove using strong induction that $x^n + \frac{1}{x^n} \in \mathbb{Z}$ for all $n \geq 0$.
Hint: More than one base case may be useful. Consider the product $(x^n + \frac{1}{x^n})(x + \frac{1}{x})$.

SCRATCH

SCRATCH

SCRATCH