17. Planar Sliding

Mechanics of Manipulation

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

Outline.

Motivation and overview.

Force and moment of planar sliding.

The limit surface.

Planar sliding

Examples:

In a gripper. Immobilization; compliant motion.

Workholding on a workbench.

Adjusting position on a workbench.

Handling large or small things.

Handling things in bulk.

Grasping and placing.

What would we like to know?

Fundamental mechanics: relation of motion to reaction forces.

Given additional applied force or kinematic constraint, predict motion.

Given desired motion, find required applied forces, constraints, initial velocity.

Examples:

Push the table.

Push a block into the corner.

Pressure distribution is key

Frictional forces are distributed over support surface.

Coulomb's law gives each frictional force a weight proportional to pressure.

Pressure is generally underdetermined.

Classify problems by pressure assumptions:

- Determined:
  - Tactile sensor;
  - One, two, or three contact points balancing known weight;
  - Linear (elastic layer between two flat rigid bodies) balancing known weight;
- Underdetermined:
  - Known support region;
  - Balancing known weight;

Force and moment of planar sliding

Let R be support region.

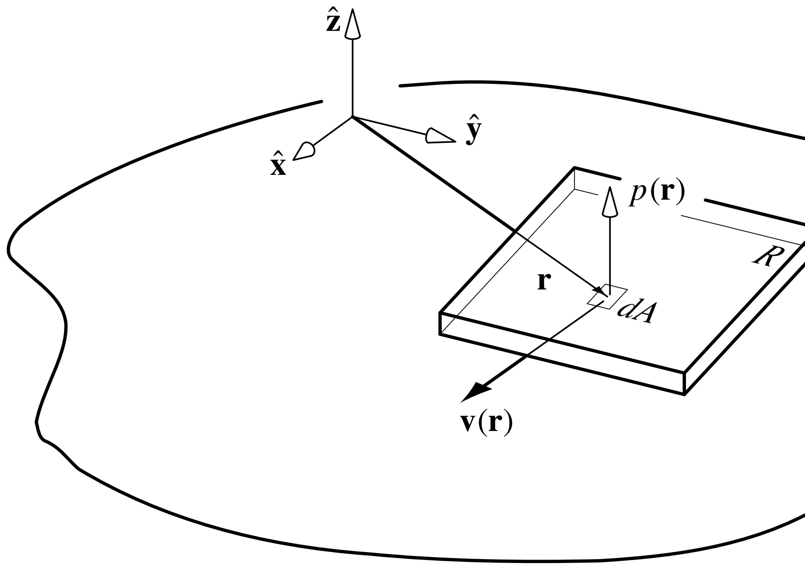
Let $\mathbf{v}(\mathbf{r})$ be vel of some point $\mathbf{r} \in R$.

Let $p(\mathbf{r})$ be the pressure at $\mathbf{r}$,

Let dA be element of area at $\mathbf{r}$,

Then normal force at $\mathbf{r}$ is given by

$$p(\mathbf{r})\, dA$$



Assume μ uniform over R.

Coulomb's law at $\mathbf{r}$:

$$-\mu \, \frac{\mathbf{v}(\mathbf{r})}{|\mathbf{v}(\mathbf{r})|} p(\mathbf{r})\, dA \qquad (1)$$

for $|\mathbf{v}(\mathbf{r})| \neq 0$.

Integrating. . .

To obtain total force and moment we integrate over R:

$$\mathbf{f}_f = -\mu \int_R \frac{\mathbf{v}(\mathbf{r})}{|\mathbf{v}(\mathbf{r})|} p(\mathbf{r})\, dA$$

$$\mathbf{n}_f = -\mu \int_R \mathbf{r} \times \frac{\mathbf{v}(\mathbf{r})}{|\mathbf{v}(\mathbf{r})|} p(\mathbf{r})\, dA$$

For known pressure distribution we can evaluate the integrals. Generally we cannot.

Force and moment for translation

Although the integrals cannot be evaluated, they can be simplified for translational sliding.

Assume translation: $\mathbf{v}(\mathbf{r}) = \mathbf{v}$.

We can pull velocity out of integral:

$$\mathbf{f}_f = -\mu \frac{\mathbf{v}}{|\mathbf{v}|} \int_R p(\mathbf{r})\, dA$$

$$\mathbf{n}_f = -\mu \int_R \mathbf{r}\, p(\mathbf{r})\, dA \times \frac{\mathbf{v}}{|\mathbf{v}|}$$

Center of friction

Let $\mathbf{f}_0$ be the total normal force:

$$f_0 = \int_R p(\mathbf{r})\, dA$$

Let $\mathbf{r}_0$ be the centroid of the pressure distribution.

$$\mathbf{r}_0 = \frac{1}{f_0} \int_R \mathbf{r}\, p(\mathbf{r})\, dA$$

Substituting above:

$$\mathbf{f}_f = -\mu \frac{\mathbf{v}}{|\mathbf{v}|} f_0$$

$$\mathbf{n}_f = \mathbf{r}_0 \times \mathbf{f}_f$$

This is Coulomb's law for a point contact! Call $\mathbf{r}_0$ the *center of friction*. In translation, frictional force acts through center of friction.

Rotation

Let $\mathbf{r}_{\mathrm{IC}}$ be the instantaneous center of rotation

Velocity at $\mathbf{r}$ is:

$$\begin{aligned}\mathbf{v}(\mathbf{r}) &= \omega \times (\mathbf{r} - \mathbf{r}_{\mathrm{IC}}) \\ &= \dot{\theta}\,\hat{\mathbf{k}} \times (\mathbf{r} - \mathbf{r}_{\mathrm{IC}})\end{aligned}$$

Direction of motion at $\mathbf{r}$ is:

$$\frac{\mathbf{v}(\mathbf{r})}{|\mathbf{v}(\mathbf{r})|} = \mathrm{sgn}(\dot{\theta})\,\hat{\mathbf{k}} \times \frac{\mathbf{r} - \mathbf{r}_{\mathrm{IC}}}{|\mathbf{r} - \mathbf{r}_{\mathrm{IC}}|}$$

Substituting

$$\begin{aligned}\mathbf{f}_f &= -\mu\ \mathrm{sgn}(\dot{\theta})\,\hat{\mathbf{k}} \times \int_R \frac{\mathbf{r} - \mathbf{r}_{\mathrm{IC}}}{|\mathbf{r} - \mathbf{r}_{\mathrm{IC}}|} p(\mathbf{r})\,dA \\ n_{fz} &= -\mu\ \mathrm{sgn}(\dot{\theta}) \int_R \mathbf{r} \cdot \frac{\mathbf{r} - \mathbf{r}_{\mathrm{IC}}}{|\mathbf{r} - \mathbf{r}_{\mathrm{IC}}|} p(\mathbf{r})\,dA\end{aligned}$$

Force to motion mapping

Define **frictional load** to be wrench applied by slider to planar support. This is a change of sign.

The integrals give the frictional load wrench as a function of slider velocity twist.

Consider the relation of frictional load to slider velocity for a finite number of support points. We will see that the motion-force mapping is neither one-to-many nor many-to-one.

The Limit Surface is an elegant geometrical representation of the motion-force mapping.

One point of support

Let $\mathbf{v}$ be the velocity of the particle.

Let $\mathbf{f}$ be the frictional load.

Coulomb's law can be stated:

slip: $\mathbf{f} \parallel \mathbf{v}$, and $|\mathbf{f}| = \mu f_n$, where μ is the coefficient of friction, and f_n is the support force.

stick: $|\mathbf{f}| \leq \mu f_n$.

Limit Curve: maximum power inequality

Consider the set of all possible frictional loads, with f_n fixed. That is a disk centered at the origin of radius μf_n.

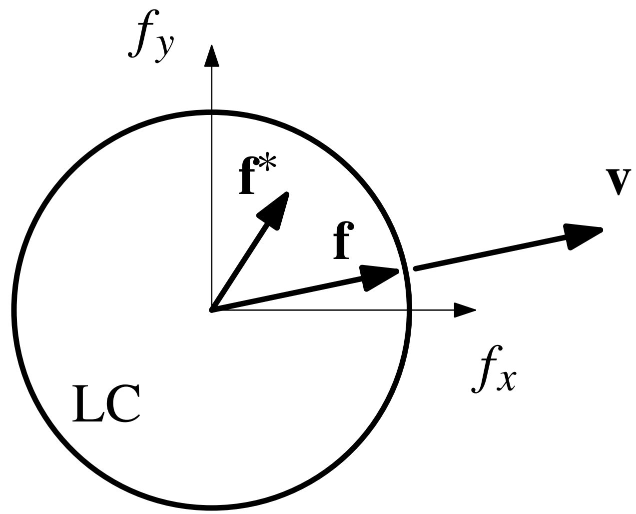
Define the **limit curve LC** to be the circle of radius μf_n at the origin of force space.

Coulomb's law is equivalent to the **maximum power inequality**:

$$\forall_{\mathbf{f}^* \in \mathrm{LC}} \quad (\mathbf{f} - \mathbf{f}^*) \cdot \mathbf{v} \geq 0$$

I.e. motion $\mathbf{v}$ yields a load that is extremal in the $\mathbf{v}$ direction.

When slip occurs $\mathbf{f}$ is on the limit curve, and $\mathbf{v}$ is *normal* to the limit curve at $\mathbf{f}$.



More than one point

Let $\mathbf{r}$ vary over the support region

Construct a limit curve $\mathrm{LC}(\mathbf{r})$ at each point $\mathbf{r}$,

At each point $\mathbf{r}$ the maximum power inequality holds:

$$\forall_{\mathbf{f}^*(\mathbf{r}) \in \mathrm{LC}(\mathbf{r})} \;\; (\mathbf{f}(\mathbf{r}) - \mathbf{f}^*(\mathbf{r})) \cdot \mathbf{v}(\mathbf{r}) \geq 0$$

How do we put them all together??? Wrenches and twists!

Frictional load wrench and velocity twist

Let $\mathbf{p}$ be the total frictional load wrench

$$\mathbf{p} = \begin{pmatrix} f_x \\ f_y \\ n_{0z} \end{pmatrix} = \sum_{\mathbf{r}} \begin{pmatrix} f_x(\mathbf{r}) \\ f_y(\mathbf{r}) \\ \mathbf{r} \times \mathbf{f}(\mathbf{r}) \end{pmatrix}$$

Let $\mathbf{q}$ be the velocity twist

$$\mathbf{q} = \begin{pmatrix} v_{0x} \\ v_{0y} \\ \omega_z \end{pmatrix}$$

Power of Coulomb load versus arbitrary load

Let $\mathbf{f}(\mathbf{r})$ be a distribution of frictional loads satisfying Coulomb's law

Let $\mathbf{f}^*(\mathbf{r})$ be arbitrary, except that at each $\mathbf{r}$, $\mathbf{f}^*(\mathbf{r})$ is in the corresponding limit curve:

$$\forall_r \;\; \mathbf{f}^*(\mathbf{r}) \in \mathrm{LC}(\mathbf{r})$$

Let $\mathbf{p}$ and $\mathbf{p}^*$ be the total frictional load wrench for $\mathbf{f}(\mathbf{r})$ and $\mathbf{f}^*(\mathbf{r})$ respectively.

The power dissipated by the Coulomb load can be written two ways

$$\mathbf{p} \cdot \mathbf{q} = \sum_{\mathbf{r}} \mathbf{f}(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r})$$

Similarly for the arbitrary load can write

$$\mathbf{p}^* \cdot \mathbf{q} = \sum_{\mathbf{r}} \mathbf{f}^*(\mathbf{r}) \cdot \mathbf{v}(\mathbf{r})$$

Power difference between Coulomb and arbitrar

Taking the difference yields

$$(\mathbf{p}-\mathbf{p}^*)\cdot\mathbf{q}=\sum_{\mathbf{r}}(\mathbf{f}(\mathbf{r})-\mathbf{f}^*(\mathbf{r}))\cdot\mathbf{v}(\mathbf{r})$$

By maximum power inequality, every term in the sum on the right hand side is non-negative. So:

$$(\mathbf{p}-\mathbf{p}^*)\cdot\mathbf{q}\geq 0$$

Another maximum power inequality! This time in wrench space!!!

Summary: Consider all loads such that $\mathbf{f}^*(\mathbf{r})\in \mathrm{LC}(\mathbf{r})$. The correct load maximizes the power dissipated.

Limit Surface

No motion: $\mathbf{q} = 0$, power dissipated is zero for any frictional load.

So, form the set of all possible total frictional load wrenches $\mathbf{p}^*$

Define the **limit surface** to be the surface of this set.

Maximum power inequality: the frictional load wrench yields maximum power over all wrenches in the limit surface.

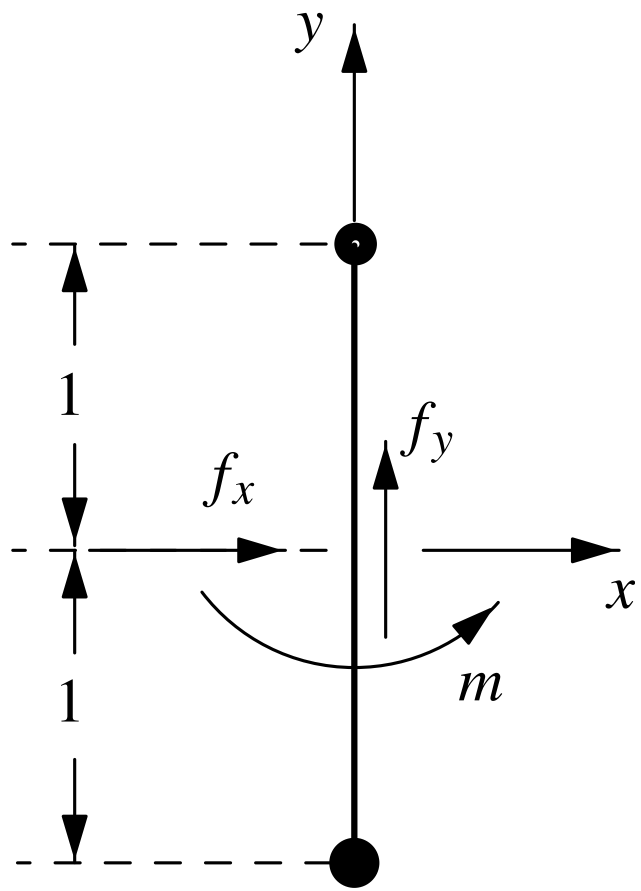
Equivalently: during slip the total frictional load wrench $\mathbf{p}$ lies on the limit surface, and the velocity twist $\mathbf{q}$ is normal to the limit surface at $\mathbf{p}$.

Example: two point contact

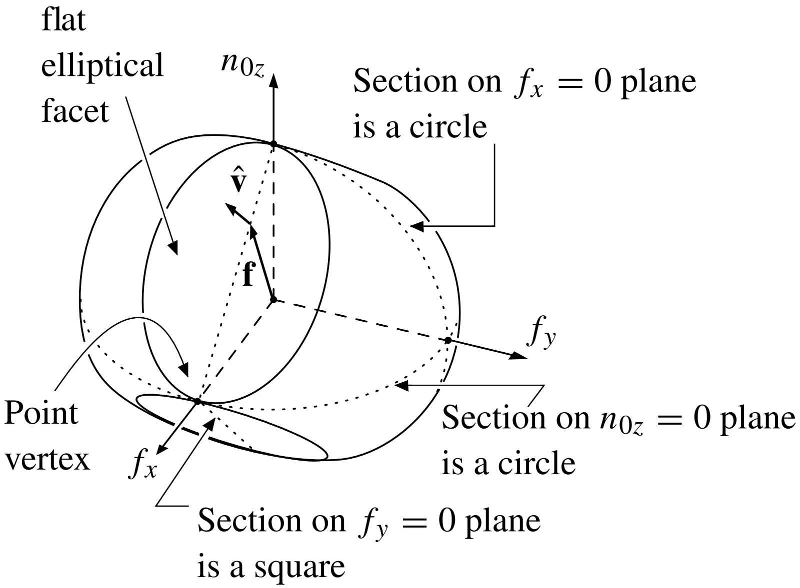
Two points of contact, support evenly divided.

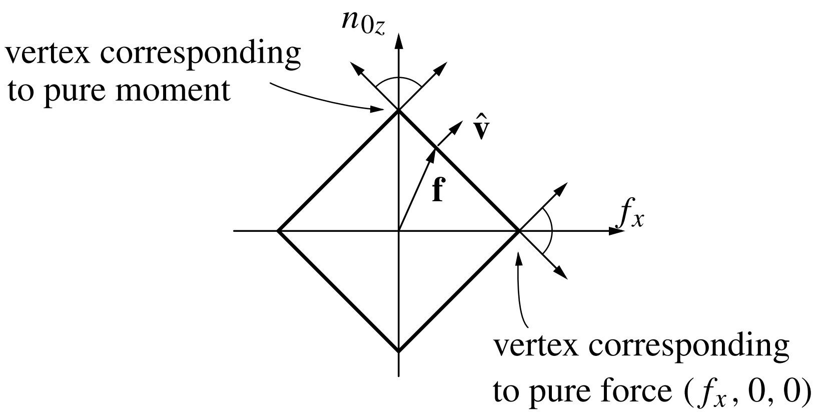
1. LS_a: all frictional loads at a. An elliptical disk in wrench space.
2. LS_b: all frictional loads at b. Also an elliptical disk.
3. The desired limit surface LS is the Minkowski sum:

$$\mathrm{LS} = \{\mathbf{w}_a + \mathbf{w}_b \mid \mathbf{w}_a \in \mathrm{LS}_a, \mathbf{w}_b \in \mathrm{LS}_b\}$$



Barbell Limit Surface





LS properties

The barbell LS illustrates some properties that hold generally: Closed, convex, enclosing the origin of wrench space.

Symmetric when reflected through origin.

Orthogonal projection onto the f_x,f_y plane is a circle of radius $\sum \mu f_n$.

Each discrete point of support yields two antipodal flat facets. On each facet several loads map to one motion (rotation about the support point.)

(No discrete points: LS is strictly convex and load-motion mapping is one-to-one.)

Collinear discrete support is even weirder: vertices on LS where one load maps to several velocities (rotation about point collinear with support).