Problem for Assignment 2: Clustering in Networks

Part A. Consider a network of \( N \) green and \( N \) purple nodes. The probability that there is a link between nodes of the same color is \( p_{in} \) and the probability that there is a link between nodes of different colors is \( p_{out} \). A network has associative clusters if \( p_{in} > p_{out} \) capturing a tendency to connect to nodes with the same color. For \( p_{out} = 0 \) the network has at least two components, containing nodes with the same color.

a. Calculate the average degree of the purple cluster made of only purple nodes, and the average degree in the entire network. \textbf{Max score: 5pts}

b. Determine the minimal \( p_{in} \) and \( p_{out} \) required to have, with high probability, just one component. \textbf{Max score: 8pts}

c. Show that for large \( N \) even very snobbish networks \( (p_{in} \gg p_{out}) \) display the small-world property. \textbf{Max score: 8pts}

Part B. Consider the following balanced variant of the above model, in which we have total \( 2N \) nodes and three clusters, two clusters of equal size containing green and purple nodes and the third cluster with fraction \( f \) of all nodes, with gray nodes. Green and purple nodes do not connect to each other (their \( p_{out} = 0 \)) while they connect with probability \( p_{in} \) to nodes of the same color. Gray nodes connect with the same probability \( q_{out} = p_{in} \) to green nodes and purple nodes, but not to any of gray nodes (their \( q_{in} = 0 \)).

a. We call the green and purple clusters \textit{interactive} if a typical purple node is just two steps away from a green node and vice versa. Evaluate the fraction of gray nodes required for the clusters to be interactive. \textbf{Max score: 8pts}

b. Comment on the size of the gray cluster if the average degree of green (or purple) nodes is \( \langle k \rangle \geq 1 \). \textbf{Max score: 8pts}

c. Discuss the implications of this model for the structure of social (and other) networks. \textbf{Max score: 8pts}