A giant cluster exists if each node is connected to at least two other nodes.

The average degree of a node i linked to the GC, must be 2, i.e.

\[ <k_m | i \leftrightarrow j >= \sum_{k_m} k_m P(k_m | i \leftrightarrow j) = 2 \]

\[ P(k_m | i \leftrightarrow j) = \frac{P(k_m, i \leftrightarrow j)}{P(i \leftrightarrow j)} = \frac{P(i \leftrightarrow j | k_m)P(k_m)}{P(i \leftrightarrow j)} \]  

Bayes’ theorem

\[ P(k_m|i \leftrightarrow j): \text{joint probability that a node has degree } k_m \text{ and is connected to nodes } i \text{ and } j. \]

For a randomly connected network (does NOT mean random network!) with \( P(k) \):

\[ P(i \leftrightarrow j) = \frac{2L}{N(N-1)} = \frac{<k>}{N-1} \]

\[ P(i \leftrightarrow j | k_m) = \frac{k_m}{N-1} \]

\[ \sum_{k_m} k_m P(k_m | i \leftrightarrow j) = \sum_{k_m} k_m \frac{P(i \leftrightarrow j | k_m)P(k_m)}{P(i \leftrightarrow j)} = \sum_{k_m} k_m P(k_m) \frac{k_m}{<k>} = \frac{\sum k_m^2 P(k_m)}{<k>} \]

\[ \kappa \equiv \frac{<k^2>}{<k>} = 2 \]

\[ \kappa > 2: \text{ a giant cluster exists} \]

\[ \kappa < 2: \text{ many disconnected clusters} \]

Apply the Malloy-Reed Criteria to an Erdos-Renyi Network

**Discrete Formulation**
- binomial distribution -

\[ P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k} \]

**Continuum Formulation**
- Poisson distribution -

\[ P(k) = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \]

**Probability Distribution Function (PDF)**

\[ \langle k \rangle = (N-1)p \]
\[ \langle k^2 \rangle = p(1-p)(N-1) + p^2(N-1)^2 \]
\[ \sigma_k = (\langle k^2 \rangle - \langle k \rangle^2)^{1/2} = [p(1-p)(N-1)]^{1/2} \]

\[ \langle k^2 \rangle = \langle k \rangle (1 + \langle k \rangle) \]
\[ \sigma_k = (\langle k^2 \rangle - \langle k \rangle^2)^{1/2} = \langle k \rangle^{1/2} \]
A giant cluster exists if each node is connected to at least two other nodes.

\[ \kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = 2 \]

\( \kappa > 2 \): a giant cluster exists;

\( \kappa < 2 \): many disconnected clusters;

\[ \langle k \rangle = \langle k \rangle \\
\langle k^2 \rangle = \langle k \rangle (1 + \langle k \rangle) \]

\[ \sigma_k = (\langle k^2 \rangle - \langle k \rangle^2)^{1/2} = \langle k \rangle^{1/2} \]

Random Network: Damage is modeled as an inverse percolation process.

Component structure

Graph

\[ \kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2 \]

\( f = \) fraction of removed nodes

(Inverse percolation phase transition)
FINAL REMARKS: EFFECT OF ASSORTATIVE MIXING: PERCOLATION

**Problem:** What are the consequences of removing a fraction $f$ of all nodes?

At what threshold $f_c$ will the network fall apart (no giant component)?

Random node removal changes

- the degree of individual nodes $[k \rightarrow k' \leq k]$
- the degree distribution $[P(k) \rightarrow P'(k')]$

A node with degree $k$ will lose some links and become a node with degree $k'$ with probability:

$$P'(k') = \sum_{k'=k}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^k$$

Let us assume that we know $<k>$ and $<k^2>$ for the original degree distribution $P(k)$

$\Rightarrow$ calculate $<k'>$, $<k'^2>$ for the new degree distribution $P'(k')$.

\[ P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'} \]

Degree distribution after we removed \( f \) fraction of nodes.

\[ \langle k' \rangle_f = \sum_{k'=0}^{\infty} k' P'(k') = \sum_{k'=0}^{\infty} k' \sum_{k=k'}^{\infty} P(k) \frac{k!}{k'!(k-k')!} f^{k-k'} (1-f)^{k'} = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} (1-f) \]

The sum is done over the triangle shown in the right, so we can replace it with

\[ \langle k' \rangle_f = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} (1-f) = \sum_{k=0}^{\infty} (1-f)kP(k) \sum_{k'=0}^{\infty} \frac{(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} \]

\[ \sum_{k=0}^{\infty} (1-f)kP(k) \sum_{k'=0}^{\infty} \binom{k}{k'-1} f^{k-k'} (1-f)^{k'-1} = \sum_{k=0}^{\infty} (1-f)kP(k) = (1-f) \langle k \rangle \]

Degree distribution after we removed f fraction of nodes.

\[ P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'} \]

\[ < k'^2 >_f = < k'(k'-1) - k' >_f = \sum_{k'=0}^{\infty} k'(k'-1) P'(k') - < k' >_f \]

The sum is done over the triangle shown in the right, i.e. we can replace it with

\[ \sum_{k'=0}^{\infty} \sum_{k=0}^{k'} = \sum_{k=0}^{\infty} \sum_{k'=0}^{\infty} \]

\[ < k'(1-k') >_f = \sum_{k'-0}^{\infty} \sum_{k=0}^{k'} P(k) \frac{k(k-1)(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k-k'} (1-f)^2 = \sum_{k=0}^{\infty} (1-f)^2 k(k-1) P(k) \sum_{k'=0}^{\infty} \frac{(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k-k'} = \sum_{k=0}^{\infty} (1-f)^2 k(k-1) P(k) = (1-f)^2 < k(k-1) > \]

\[ < k'^2 >_f = < k'(k'-1) - k' >_f = (1-f)^2 (< k'^2 > - < k >) - (1-f) < k > = (1-f)^2 < k^2 > + f(1-f) < k > \]

**Robustness:** we remove a fraction $f$ of the nodes.

At what threshold $f_c$ will the network fall apart (no giant component)?

Random node removal changes
- the degree of individuals nodes $[k \rightarrow k' \leq k)$
- the degree distribution $[P(k) \rightarrow P'(k')]$

Breakdown threshold:

$$f_c = 1 - \frac{1}{\langle k^2 \rangle / \langle k \rangle - 1}$$

$0 < f < f_c$: the network is still connected (there is a giant cluster)

$f > f_c$: the network becomes disconnected (giant cluster vanishes)

Scale-free networks do not appear to break apart under random failures. Reason: the hubs. The likelihood of removing a hub is small.

Scale-free networks do not appear to break apart under random failures. Why is that?

\[ < k^m > = (\gamma - 1) K_{\text{min}}^{\gamma - 1} \int k^{m-\gamma} dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} K_{\text{min}}^{\gamma - 1} K_{\text{max}}^{m - \gamma + 1} K_{\text{min}}^{m - \gamma + 1} \]

\[ K_{\text{max}} = K_{\text{min}} N^{\gamma - 1} \]

\[ f_c = 1 - \frac{1}{\left< k^2 \right>} - 1 \]

\[ \frac{< k^2 >}{< k >} = \frac{(2 - \gamma) K_{\text{max}}^{3 - \gamma} - K_{\text{min}}^{3 - \gamma}}{(3 - \gamma) K_{\text{max}}^{2 - \gamma} - K_{\text{min}}^{2 - \gamma}} \]

\[ \gamma > 3: \quad \kappa = \frac{(2 - \gamma) K_{\text{max}}^{3 - \gamma} - K_{\text{min}}^{3 - \gamma}}{(3 - \gamma) K_{\text{max}}^{2 - \gamma} - K_{\text{min}}^{2 - \gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\text{min}} \]

\[ 3 > \gamma > 2: \quad \kappa = \frac{(2 - \gamma) K_{\text{max}}^{3 - \gamma} - K_{\text{min}}^{3 - \gamma}}{(3 - \gamma) K_{\text{max}}^{2 - \gamma} - K_{\text{min}}^{2 - \gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\text{max}}^{3 - \gamma} K_{\text{min}}^{\gamma - 2} \]

\[ 2 > \gamma > 1: \quad \kappa = \frac{(2 - \gamma) K_{\text{max}}^{3 - \gamma} - K_{\text{min}}^{3 - \gamma}}{(3 - \gamma) K_{\text{max}}^{2 - \gamma} - K_{\text{min}}^{2 - \gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\text{max}}^{3 - \gamma} K_{\text{min}}^{\gamma - 2} \]
\[ f_c = 1 - \frac{1}{\kappa - 1} \]

\[ \kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = \begin{cases} \frac{2 - \gamma}{3 - \gamma} & \gamma > 3 \\ \frac{K_{\text{min}}}{K_{\text{max}}} & 3 > \gamma > 2 \\ \frac{1}{K_{\text{max}}} & 2 > \gamma > 1 \end{cases} \]

\[ K_{\text{max}} = K_{\text{min}} N^{\gamma^{-1}} \]

\(\gamma > 3\): \(\kappa\) is finite, so the network will break apart at a finite \(f_c\) that depends on \(K_{\text{min}}\)

\(\gamma < 3\): \(\kappa\) diverges in the \(N \to \infty\) limit, so \(f_c \to 1\) !!!

for an infinite system one needs to remove all the nodes to break the system.

For a finite system, there is a finite but large \(f_c\) that scales with the system size as:

\[ \kappa \approx 1 - C N^{\frac{3-\gamma}{\gamma-1}} \]

**Internet**: Router level map, \(N=228,263; \gamma=2.1 \pm 0.1; \kappa=28 \to f_c = 0.962\)

**AS level map**, \(N=11,164; \gamma=2.1 \pm 0.1; \kappa=264 \to f_c = 0.996\)
Scale-free random graph with

\[ P(k) = Ak^{-\gamma}, \text{ with } k = m, \ldots, K \]

\[ f_c = 1 - \frac{1}{\gamma - 2} \frac{m - 1}{m - 1} \quad \text{if } \gamma > 3 \]

\[ f_c = 1 - \frac{1}{\gamma - 2} \frac{m^{\gamma - 2}K^{3-\gamma} - 1}{3 - \gamma} \quad \text{if } 2 < \gamma < 3 \]


Infinite scale-free networks with \( \gamma < 3 \) do not break down under random node failures.
$S$: size of the giant component, $f$ fraction of randomly removed nodes, not damage for $f<f_c$

(i) \( \gamma>4: \ S \approx f-f_c \) (similar to that of a random graph)

(ii) \( 3>\gamma>4: \ S \approx (f-f_c)^{1/(\gamma-3)} \)

(iii) \( \gamma<3: \ f_c = 0 \) and \( S \approx f^{1+1/(3-\gamma)} \)

R. Cohen, D. ben-Avraham, S. Havlin,
Percolation critical exponents in scale-free networks
Phys. Rev. E 66, 036113 (2002);
See also: Dorogovtsev S, Lectures on Complex Networks, Oxford, pg44
Achilles’ Heel of scale-free networks

\( \gamma \leq 3 : f_c = 1 \)

(R. Cohen et al PRL, 2000)

INTERNET’S ROBUSTNESS TO RANDOM FAILURES

\[ f_c = 1 - \frac{1}{\kappa - 1} \]

Internet: Router level map, N=228,263; $\gamma=2.1 \pm 0.1$; $\kappa=28 \quad \Rightarrow \quad f_c = 0.962$

AS level map, N= 11,164; $\gamma=2.1 \pm 0.1$; $\kappa=264 \quad \Rightarrow \quad f_c = 0.996$

Internet parameters: Pastor-Satorras & Vespignani, Evolution and Structure of the Internet: Table 4.1 & 4.4

Attack threshold for arbitrary $P(k)$

**Attack problem**: we remove a fraction $f$ of the hubs.

At what threshold $f_c$ will the network fall apart (no giant component)?

Hub removal changes

- the maximum degree of the network [$K_{\text{max}} \rightarrow K'_{\text{max}} \leq K_{\text{max}}$]
- the degree distribution [$P(k) \rightarrow P'(k')$]

A node with degree $k$ will lose some links because some of its neighbors will vanish.

Claim: once we correct for the changes in $K_{\text{max}}$ and $P(k)$, we are back to the robustness problem. That is, attack is nothing but a robustness of the network with a new $K_{\text{max}}$ and $P(k)$.

Attack problem: we remove a fraction $f$ of the hubs.

the maximum degree of the network $[K_{\text{max}} \rightarrow K'_{\text{max}} \leq K_{\text{max}}]$. 

If we remove an $f$ fraction of hubs, the maximum degree changes:

$$\int P(k)dk = f$$

$$\int P(k)dk = (γ - 1)K_{\text{min}}^{γ - 1} \int k^{-γ}dk = \frac{γ - 1}{1 - γ} K_{\text{min}}^{γ - 1} (K_{\text{max}}^{1 - γ} - K'_{\text{max}}^{1 - γ})$$

As $K'_{\text{max}} \leq K_{\text{max}}$ we can ignore the $K_{\text{max}}$ term

$$\left(\frac{K_{\text{min}}}{K'_{\text{max}}}\right)^{γ - 1} = f$$

$$K'_{\text{max}} = K_{\text{min}} f^{1 - γ}$$

$\leftrightarrow$ The new maximum degree after removing $f$ fraction of the hubs.

Attack threshold for arbitrary $P(k)$

**Attack problem**: we remove a fraction $f$ of the hubs.

the degree distribution changes $[P(k) \rightarrow P'(k')]$

A node with degree $k$ will lose some links because some of its neighbors will vanish.

Let us calculate the fraction of links removed ‘randomly’, $f'$, as a consequence of we removing $f$ fraction of hubs.

$$f' = \frac{\int_0^{\kappa_{\text{max}}} kP(k)dk}{\int_0^{\kappa_{\text{max}}} k^{1-\gamma}P(k)dk} = \frac{1}{<k> (\gamma - 1) \kappa_{\text{min}}^{\gamma-1} \int_0^{\kappa_{\text{max}}} k^{1-\gamma}dk} = \frac{1}{<k> 2-\gamma} \kappa_{\text{min}}^{\gamma-1}(K_{\text{max}}^{2-\gamma} - K_{\text{min}}^{2-\gamma})$$

For $\gamma \rightarrow 2$, $f' \rightarrow 1$, which means that even the removal of a tiny fraction of hubs will destroy the network. The reason is that for $\gamma=2$ hubs dominate the network.

Attack threshold for arbitrary P(k)

**Attack problem:** we remove a fraction $f$ of the hubs.

At what threshold $f_c$ will the network fall apart (no giant component)?

Hub removal changes

- the maximum degree of the network $[K_{\text{max}} \rightarrow K'_{\text{max}} \leq K_{\text{max}}]$  
  $K'_{\text{max}} = K_{\text{min}} f^{1-\gamma}$

- the degree distribution $[P(k) \rightarrow P'(k')]$

A node with degree $k$ will lose some links because some of its neighbors will vanish. $f' = f^{1-\gamma}$

Claim: once we correct for the changes in $K_{\text{max}}$ and $P(k)$, we are back to the robustness problem. That is, attack is nothing but a robustness of the network with a new $K'_{\text{max}}$ and $f'$.

\[
f' = 1 - \frac{1}{\kappa' - 1}
\]

\[
\kappa' = \frac{\langle k'^2 \rangle}{\langle k' \rangle} = \frac{\langle k^2 \rangle}{(1 - f_c) \langle k \rangle} = \frac{\kappa}{1 - f_c}
\]

\[
\kappa = \begin{cases} 
\frac{2 - \gamma}{3 - \gamma} \left[ \frac{K_{\text{min}}}{K_{\text{max}}} \right]^{\gamma - 2} & \gamma > 3 \\
\frac{3 - \gamma}{2 - \gamma} K_{\text{min}}^{\gamma - 2} & 3 > \gamma > 2 \\
K_{\text{max}} & 2 > \gamma > 1 
\end{cases}
\]

\[
f_c^{2-\gamma} = 2 + \frac{2 - \gamma}{3 - \gamma} K_{\text{min}} \left( f_c^{3-\gamma} - 1 \right)
\]

**Attack threshold for arbitrary P(k)**

**Attack problem:** we remove a fraction $f$ of the hubs.
At what threshold $f_c$ will the network fall apart (no giant component)?

\[
f_c^{\frac{2-\gamma}{1-\gamma}} = 2 + \frac{2-\gamma}{3-\gamma} K_{\text{min}} \left( f_c^{\frac{3-\gamma}{1-\gamma}} - 1 \right)
\]

- $f_c$ depends on $\gamma$; it reaches its max for $\gamma<3$
- $f_c$ depends on $K_{\text{min}}$ (m in the figure)
- Most important: $f_c$ is tiny. Its maximum reaches only 6%, i.e. the removal of 6% of nodes can destroy the network in an attack mode.
- Internet: $\gamma=2.1$, so 4.7% is the threshold.

Figure: Pastor-Satorras & Vespignani, *Evolution and Structure of the Internet*: Fig 6.12

Application: ER random graphs

Consider a random graph with connection probability $p$ such that at least a giant connected component is present in the graph.

Find the critical fraction of removed nodes such that the giant connected component is destroyed.

$$f_c = 1 - \frac{1}{\langle k_0^2 \rangle} = 1 - \frac{1}{pN} = 1 - \frac{1}{\langle k_0 \rangle}$$

The higher the original average degree, the larger damage the network can survive.

Q: How do you explain the peak in the average distance?
Summary: Achilles’ Heel of scale-free networks

Network Science: Robustness Cascades

(R. Cohen et al PRL, 2000)

Summary: Achilles’ Heel of complex networks

Internet

A network of n-ary degree of connectivity has n links per node was simulated.

The simulation revealed that networks where $n \geq 3$ had a significant increase in resilience against even as much as 50% node loss. Baran's insight gained from the simulation was that redundancy was the key.
Scale-free networks are more error tolerant, but also more vulnerable to attacks

- squares: random failure
- circles: targeted attack

**Failures:** little effect on the integrity of the network.
**Attacks:** fast breakdown
Real scale-free networks show the same dual behavior

- blue squares: random failure
- red circles: targeted attack
- open symbols: $S$
- filled symbols: $l$

- break down if 5% of the nodes are eliminated selectively (always the highest degree node)
- resilient to the random failure of 50% of the nodes.

Similar results have been obtained for metabolic networks and food webs.

Network Science: Robustness Cascades
Cascades

Potentially large events triggered by small initial shocks

- **Information cascades**
  social and economic systems
diffusion of innovations

- **Cascading failures**
  infrastructural networks
complex organizations
Cascading Failures in Nature and Technology

Flows of physical quantities
- congestions
- instabilities
- Overloads

Cascades depend on
- Structure of the network
- Properties of the flow
- Properties of the net elements
- Breakdown mechanism
Northeast Blackout of 2003

Origin
A 3,500 MW power surge (towards Ontario) affected the transmission grid at 4:10:39 p.m. EDT. (Aug-14-2003)

Consequences
More than 508 generating units at 265 power plants shut down during the outage. In the minutes before the event, the NYISO-managed power system was carrying 28,700 MW of load. At the height of the outage, the load had dropped to 5,716 MW, a loss of 80%.
Cascading disaster in Japan

Blast shakes a second reactor, death toll soars

By Martin Fackler
and Mark McDonald
NEW YORK TIMES

SENDAI, Japan — Japan reeled from a rapidly unfolding disaster of epic scale yesterday, pummeled by a death toll, destruction, and homelessness caused by the earthquake and tsunami and new hazards from damaged nuclear reactors. The prime minister called it Japan's worst crisis since World War II.

Japan's $5 trillion economy, the world's third largest, was threatened with severe disruptions and partial paralysis as many industries shut down temporarily. The armed forces and volunteers mobilized for the far more urgent crisis of finding survivors, evacuating residents near the stricken power plants and caring for the victims of the record 8.9 magnitude quake that struck on Friday.

The disaster has left more than 10,000 people homeless, and millions without water, power, heat, or transportation.
Cascades Size Distribution of Blackouts


Unserved energy/power magnitude ($S$) distribution:

$$P(S) \sim S^{-\alpha}, \; 1 < \alpha < 2$$

<table>
<thead>
<tr>
<th>Source</th>
<th>Exponent</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>North America</td>
<td>2.0</td>
<td>Power</td>
</tr>
<tr>
<td>Sweden</td>
<td>1.6</td>
<td>Energy</td>
</tr>
<tr>
<td>Norway</td>
<td>1.7</td>
<td>Power</td>
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<tr>
<td>New Zealand</td>
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<td>Energy</td>
</tr>
<tr>
<td>China</td>
<td>1.8</td>
<td>Energy</td>
</tr>
</tbody>
</table>

Cascades Size Distribution of Earthquakes

Earthquake size $S$ distribution

$P(S) \sim S^{-\alpha}, \alpha \approx 1.67$

Failure Propagation Model

Initial Setup
• Random graph with $N$ nodes
• Initially each node is functional.

Cascade
• Initiated by the failure of one node.
• $f_i$: fraction of failed neighbors of node $i$. Node $i$ fails if $f_i$ is greater than a global threshold $\phi$.

Erdos-Renyi network
$P(S) \sim S^{-3/2}$