Frontiers of Network Science
Fall 2018

Class 18: Robustness Part I
(Chapter 8 in Textbook)

Boleslaw Szymanski

based on slides by Albert-László Barabási and Roberta Sinatra

www.BarabasiLab.com
A giant cluster exists if each node is connected to at least two other nodes.

The average degree of a node $i$ linked to the GC, must be 2, i.e.

$$< k_m | i \leftrightarrow j > = \sum_{k_m} k_m P(k_m | i \leftrightarrow j) = 2$$

$$P(k_m | i \leftrightarrow j) = \frac{P(k_m, i \leftrightarrow j)}{P(i \leftrightarrow j)} = \frac{P(i \leftrightarrow j | k_m)P(k_m)}{P(i \leftrightarrow j)}$$

Bayes’ theorem

$P(k_m|i \leftrightarrow j)$: joint probability that a node has degree $k_m$ and is connected to nodes $i$ and $j$. For a randomly connected network (does NOT mean random network!) with $P(k)$:

$$P(i \leftrightarrow j) = \frac{2L}{N(N-1)} = \frac{< k >}{N-1}$$

$$P(i \leftrightarrow j | k_m) = \frac{k_m}{N-1}$$

$$\sum_{k_m} k_m P(k_m | i \leftrightarrow j) = \sum_{k_m} k_m \frac{P(i \leftrightarrow j | k_m)P(k_m)}{P(i \leftrightarrow j)} = \sum_{k_m} k_m \frac{k_mP(k_m)}{< k >} = \frac{\sum k_m^2 P(k_m)}{< k >}$$

$$\kappa \equiv \frac{< k^2 >}{< k >} = 2$$

$\kappa > 2$: a giant cluster exists

$\kappa < 2$: many disconnected clusters

Apply the Malloy-Reed Criteria to an Erdos-Renyi Network

**Discrete Formulation**
- binomial distribution -

\[
P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}
\]

**Continuum Formulation**
- Poisson distribution -

\[
P(k) = e^{-<k>} \frac{<k>^k}{k!}
\]

\[<k> = (N-1)p\]
\[<k>^2 = p(1-p)(N-1) + p^2(N-1)^2\]
\[\sigma_k = (<k>^2 - <k>^2)^{1/2} = [p(1-p)(N-1)]^{1/2}\]
\[<k>^2 = <k>(1+<k>)\]
\[\sigma^2 = (<k>^2 - <k>^2)^{1/2} = <k>^{1/2}\]
A giant cluster exists if each node is connected to at least two other nodes.

\[ \kappa \equiv \frac{\langle k^2 \rangle}{\langle k \rangle} = 2 \]

\( \kappa > 2 \): a giant cluster exists;

\( \kappa < 2 \): many disconnected clusters;

\[ \langle k \rangle = \langle k \rangle \]

\[ \langle k^2 \rangle = \langle k \rangle (1 + \langle k \rangle) \]

\[ \sigma_k = (\langle k^2 \rangle - \langle k \rangle^2)^{1/2} = \langle k \rangle^{1/2} \]

Component structure graph

\( f = \text{fraction of removed nodes} \)

Random network: Damage is modeled as an inverse percolation process

\( \kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = 2 \)

\( f_c = \frac{1-1/N}{\langle k \rangle} \)

\( \langle k \rangle = N-1 \)

\( k_c : \langle k \rangle = 1 \)

\( f = 1/\langle k \rangle \)

\( \langle k \rangle \to 0 \)

(Inverse percolation phase transition)
FINAL REMARKS: EFFECT OF ASSORTATIVE MIXING: PERCOLATION

**Problem:** What are the consequences of removing a fraction $f$ of all nodes?

At what threshold $f_c$ will the network fall apart (no giant component)?

Random node removal changes

- the degree of individual nodes $[k \rightarrow k' \leq k]$
- the degree distribution $[P(k) \rightarrow P'(k')]$

A node with degree $k$ will lose some links and become a node with degree $k'$ with probability:

$$\binom{k}{k'} f^{k-k'} (1-f)^k$$  \hspace{2cm} k' \leq k$$

The probability that we had a $k$ degree node was $P(k)$, so the probability that we will have a new node with degree $k'$:

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^k$$

Let us assume that we know $<k>$ and $<k^2>$ for the original degree distribution $P(k)$ → calculate $<k'>$ , $<k'^2>$ for the new degree distribution $P'(k')$.

Degree distribution after we removed $f$ fraction of nodes.

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^k$$

$$<k'>_f = \sum_{k'=0}^{\infty} k' P'(k') = \sum_{k'=0}^{\infty} k' \sum_{k=k'}^{\infty} P(k) \frac{k!}{k'!(k-k')!} f^{k-k'} (1-f)^k = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k-1} (1-f)$$

The sum is done over the triangle shown in the right, so we can replace it with

$$<k'>_f = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k-1} (1-f)$$

$$= \sum_{k=0}^{\infty} (1-f) k P(k) \sum_{k'=0}^{k} \sum_{k=0}^{k'} \frac{(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k-1} = \sum_{k=0}^{\infty} (1-f) k P(k) \sum_{k'=0}^{k} \binom{k-1}{k'-1} f^{k-k'} (1-f)^{k-1} = \sum_{k=0}^{\infty} (1-f) k P(k) = (1-f) <k>$$

\[ P'(k') = \sum_{k' - k}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^k \]

Degree distribution after we removed f fraction of nodes.

\[ <k'^2>_f = <k'(k' - 1) - k'>_f = \sum_{k' = 0}^{\infty} k'(k' - 1)P'(k') - <k'>_f \]

The sum is done over the triangle shown in the right, i.e., we can replace it with

\[ \sum_{k' = 0}^{\infty} \sum_{k = k'}^{\infty} = \sum_{k = 0}^{\infty} \sum_{k' = 0}^{k} \]

\[ <k'(1-k')>_f = \sum_{k' = 0}^{\infty} \sum_{k = k'}^{\infty} P(k) \frac{k(k-1)(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k-2} (1-f)^2 = \sum_{k = 0}^{\infty} (1-f)^2 k(k-1)P(k) \sum_{k' = 0}^{\infty} \frac{(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k-2} = \sum_{k = 0}^{\infty} (1-f)^2 k(k-1)P(k) = (1-f)^2 <k(k-1)> \]

\[ <k'^2>_f = <k'(k' - 1) - k'>_f = (1-f)^2 (k^2 - <k^2>) - (1-f) <k> = (1-f)^2 k^2 + f(1-f) <k> \]

Robustness: we remove a fraction $f$ of the nodes. At what threshold $f_c$ will the network fall apart (no giant component)?

Random node removal changes

- the degree of individuals nodes $[k \rightarrow k' \leq k]$ 
- the degree distribution $[P(k) \rightarrow P'(k')]$

\[
<k'>_f = (1 - f) < k > \\
<k'^2>_f = (1 - f)^2 < k^2 > + f(1-f) < k > \\
\kappa \equiv \frac{<k'^2>_f}{<k'>_f} = 2
\]

$\kappa > 2$: a giant cluster exists  
$\kappa < 2$: many disconnected clusters

Breakdown threshold:

\[
f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}
\]

$f < f_c$: the network is still connected (there is a giant cluster)  
$f > f_c$: the network becomes disconnected (giant cluster vanishes)

Scale-free networks do not appear to break apart under random failures. Reason: the hubs. The likelihood of removing a hub is small.

Scale-free networks do not appear to break apart under random failures. Why is that?

\[
<k^m> = (\gamma - 1)K_{\min}^{\gamma-1} \int k^{m-\gamma} dk = \frac{(\gamma - 1)}{(m - \gamma + 1)} K_{\min}^{\gamma-1} [K_{\max}^{m-\gamma+1} - K_{\min}^{m-\gamma+1}]
\]

\[
f_c = 1 - \frac{1}{\langle k^2 \rangle} \left( \frac{1}{\langle k \rangle} - 1 \right)
\]

\[
K_{\max} = K_{\min} N^{\gamma-1}
\]

\[
\gamma > 3: \quad \kappa = \frac{(2 - \gamma) K_{\max}^{3-\gamma} - K_{\min}^{3-\gamma}}{(3 - \gamma) K_{\max}^{2-\gamma} - K_{\min}^{2-\gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\min}^{3-\gamma}
\]

\[
3 > \gamma > 2: \quad \kappa = \frac{(2 - \gamma) K_{\max}^{3-\gamma} - K_{\min}^{3-\gamma}}{(3 - \gamma) K_{\max}^{2-\gamma} - K_{\min}^{2-\gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\max}^{3-\gamma} K_{\min}^{\gamma-2}
\]

\[
2 > \gamma > 1: \quad \kappa = \frac{(2 - \gamma) K_{\max}^{3-\gamma} - K_{\min}^{3-\gamma}}{(3 - \gamma) K_{\max}^{2-\gamma} - K_{\min}^{2-\gamma}} = \frac{2 - \gamma}{3 - \gamma} K_{\max}
\]
\[ f_c = 1 - \frac{1}{\kappa - 1} \]

\[ \kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} = \begin{cases} \frac{2 - \gamma}{3 - \gamma} & \gamma > 3 \\ \frac{1}{K_{\text{max}}} N^{\gamma - 1} & \gamma < 3 \end{cases} \]

\[ K_{\text{max}} = K_{\text{min}} N^{\gamma - 1} \]

\( \gamma > 3: \kappa \) is finite, so the network will break apart at a finite \( f_c \) that depends on \( K_{\text{min}} \)

\( \gamma < 3: \kappa \) diverges in the \( N \to \infty \) limit, so \( f_c \to 1 \) !!!

For an infinite system, one needs to remove all the nodes to break the system.

For a finite system, there is a finite but large \( f_c \) that scales with the system size as:

\[ \kappa \approx 1 - C N^{\frac{3-\gamma}{\gamma-1}} \]

**Internet**: Router level map, \( N=228,263; \gamma=2.1 \pm 0.1; \kappa=28 \quad \Rightarrow \quad f_c=0.962 \)

**AS level map**, \( N=11,164; \gamma=2.1 \pm 0.1; \kappa=264 \quad \Rightarrow \quad f_c=0.996 \)
Scale-free random graph with

\[ P(k) = Ak^{-\gamma}, \text{ with } k = m, \ldots, K \]

\[
f_c = 1 - \frac{1}{\frac{\gamma - 2}{\gamma - 3} m - 1} \text{ if } \gamma > 3
\]

\[
f_c = 1 - \frac{1}{\frac{\gamma - 2}{3 - \gamma} m^{\gamma - 2} K^{3 - \gamma} - 1} \text{ if } 2 < \gamma < 3
\]


Infinite scale-free networks with \( \gamma < 3 \) do not break down under random node failures.
S: size of the giant component, \( f \) fraction of randomly removed nodes, not damage for \( f < f_c \)

(i) \( \gamma > 4 \): \( S \approx f - f_c \) (similar to that of a random graph)

(ii) \( 3 > \gamma > 4 \): \( S \approx (f - f_c)^{1/(\gamma - 3)} \)

(iii) \( \gamma < 3 \): \( f_c = 0 \) and \( S \approx f^{1+1/(3-\gamma)} \)

Achilles’ Heel of scale-free networks

Attacks

Failures

\( \gamma \leq 3 : f_c = 1 \)

(R. Cohen et al PRL, 2000)

INTERNET’S ROBUSTNESS TO RANDOM FAILURES

\[ f_c = 1 - \frac{1}{\kappa - 1} \]

Internet: Router level map, N=228,263; \( \gamma = 2.1 \pm 0.1 \); \( \kappa = 28 \) \( \Rightarrow f_c = 0.962 \)

AS level map, N= 11,164; \( \gamma = 2.1 \pm 0.1 \); \( \kappa = 264 \) \( \Rightarrow f_c = 0.996 \)

Internet parameters: Pastor-Satorras & Vespignani, *Evolution and Structure of the Internet*: Table 4.1 & 4.4

*Network Science: Robustness Cascades*
Attack threshold for arbitrary $P(k)$

**Attack problem:** we remove a fraction $f$ of the hubs.

At what threshold $f_c$ will the network fall apart (no giant component)?

Hub removal changes

- the maximum degree of the network $[K_{\text{max}} \rightarrow K'_{\text{max}} \leq K_{\text{max}}]$.
- the degree distribution $[P(k) \rightarrow P'(k')]$.

A node with degree $k$ will lose some links because some of its neighbors will vanish.

Claim: once we correct for the changes in $K_{\text{max}}$ and $P(k)$, we are back to the robustness problem.

That is, attack is nothing but a robustness of the network with a new $K_{\text{max}}$ and $P(k)$.

Attack threshold for arbitrary $P(k)$

**Attack problem:** we remove a fraction $f$ of the hubs.

the maximum degree of the network \([K_{\text{max}} \rightarrow K'_{\text{max}} \leq K_{\text{max}}]\)

If we remove an $f$ fraction of hubs, the maximum degree changes:

\[
\int_{K'_{\text{max}}}^{K_{\text{max}}} P(k) dk = f
\]

\[
\int_{K'_{\text{max}}}^{K_{\text{max}}} P(k) dk = (\gamma - 1) K_{\text{min}}^{\gamma-1} \int_{K'_{\text{max}}}^{K_{\text{max}}} k^{-\gamma} dk = \frac{\gamma - 1}{1 - \gamma} K_{\text{min}}^{\gamma-1} (K_{\text{max}}^{1-\gamma} - K'_{\text{max}}^{1-\gamma})
\]

As $K'_{\text{max}} \leq K_{\text{max}}$ we can ignore the $K_{\text{max}}$ term

\[
\left(\frac{K_{\text{min}}}{K'_{\text{max}}}\right)^{\gamma-1} = f
\]

\[
K'_{\text{max}} = K_{\text{min}} f^{1-\gamma}
\]

\[\left(\frac{K_{\text{min}}}{K'_{\text{max}}}\right)^{\gamma-1} = f \quad \Rightarrow \text{The new maximum degree after removing } f \text{ fraction of the hubs.}\]

Attack threshold for arbitrary $P(k)$

**Attack problem:** we remove a fraction $f$ of the hubs.

The degree distribution changes $[P(k) \rightarrow P'(k')]$

A node with degree $k$ will lose some links because some of its neighbors will vanish.

Let us calculate the fraction of links removed ‘randomly’ , $f'$, as a consequence of we removing $f$ fraction of hubs.

$$f' = \frac{\int_0^{k_{\text{max}}} kP(k)dk}{\int_0^{k_{\text{max}}} kP(k)dk} = \frac{1}{<k>} (\gamma - 1)K_{\text{min}}^{\gamma-1} \int_{K_{\text{min}}}^{k_{\text{max}}} k^{1-\gamma}dk = \frac{1}{<k>^{2-\gamma}} K_{\text{min}}^{\gamma-1} (K_{\text{max}}^{2-\gamma} - K_{\text{min}}^{2-\gamma}) = -\frac{1}{<k>^{2-\gamma}} K_{\text{min}}^{\gamma-1} K_{\text{max}}^{2-\gamma}$$

For $\gamma \rightarrow 2$, $f' \rightarrow 1$, which means that even the removal of a tiny fraction of hubs will destroy the network. The reason is that for $\gamma=2$ hubs dominate the network.