Frontiers of Network Science
Fall 2018

Class 18: Robustness Part I
(Chapter 8 in Textbook)

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based on slides by
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MALLOY-REED CRITERIA: THE EXISTENCE OF A GIANT COMPONENT

A giant cluster exists if each node is connected to at least two other nodes.

The average degree of a node i linked to the GC, must be 2, i.e.

$$< k_m \mid i \leftrightarrow j >= \sum_{k_m} k_m P(k_m \mid i \leftrightarrow j) = 2$$

$$P(k_m \mid i \leftrightarrow j) = \frac{P(k_m, i \leftrightarrow j)}{P(i \leftrightarrow j)} = \frac{P(i \leftrightarrow j \mid k_m) P(k_m)}{P(i \leftrightarrow j)}$$

Bayes' theorem

$P(k_m|i <-> j)$: joint probability that a node has degree k_m and is connected to nodes i and j.

For a randomly connected network (does NOT mean random network!) with P(k):

$$P(i \leftrightarrow j) = \frac{2L}{N(N-1)} = \frac{< k >}{N-1}$$

$$P(i \leftrightarrow j \mid k_m) = \frac{k_m}{N-1}$$

***i* can choose between N-1 nodes to link to, each with probability 1/(N-1). I can try k_i times.**

$$\sum_{k_m} k_m P(k_m \mid i \leftrightarrow j) = \sum_{k_m} k_m \frac{P(i \leftrightarrow j \mid k_m) P(k_m)}{P(i \leftrightarrow j)} = \sum_{k_m} k_m \frac{k_m P(k_m)}{< k >} = \frac{\sum_{k_m} k_m^2 P(k_m)}{< k >}$$

$$\boxed{\kappa \equiv \frac{< k^2 >}{< k >} = 2}$$

κ>2: a giant cluster exists

κ<2: many disconnected clusters

Malloy, Reed, Random Structures and Algorithms (1995); Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

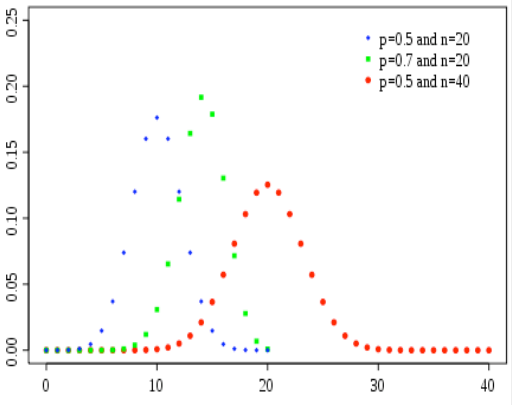
Apply the Malloy-Reed Criteria to an Erdos-Renyi Network

Probability Distribution Function (PDF)

Discrete Formulation

-binomial distribution-

$$P(k) = \binom{N-1}{k} p^k (1-p)^{(N-1)-k}$$



$$< k > = (N-1)p$$

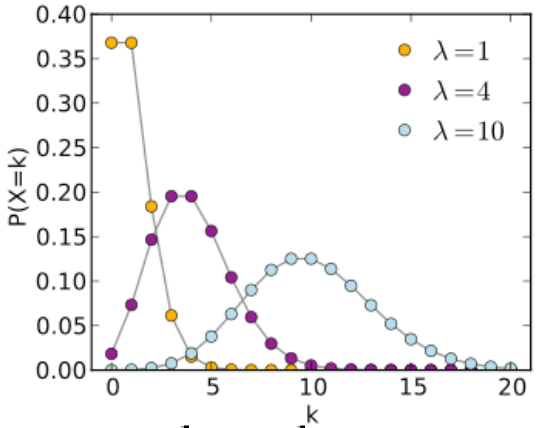
$$< k^2 > = p(1-p)(N-1) + p^2(N-1)^2$$

$$\sigma_k = (< k^2 > - < k >^2)^{1/2} = [p(1-p)(N-1)]^{1/2}$$

Continuum Formulation

-Poisson distribution-

$$P(k) = e^{-<k>} \frac{< k >^k}{k!}$$



$$< k > = < k >$$

$$< k^2 > = < k > (1 + < k >)$$

$$\sigma_k = (< k^2 > - < k >^2)^{1/2} = < k >^{1/2}$$

Apply the Malloy-Reed Criteria to an Erdos-Renyi Network

A giant cluster exists if each node is connected to at least two other nodes.

$$\kappa \equiv \frac{<k^2>}{<k>} = 2$$

κ>2: a giant cluster exists;

κ<2: many disconnected clusters;

$$<k> = <k>$$

$$<k^2> = <k>(1+<k>)$$

$$\sigma_k = (<k^2> - <k>^2)^{1/2} = <k>^{1/2}$$

$$\kappa \equiv \frac{<k^2>}{<k>} = \frac{<k>(1+<k>)}{<k>} = 1+<k> = 2$$

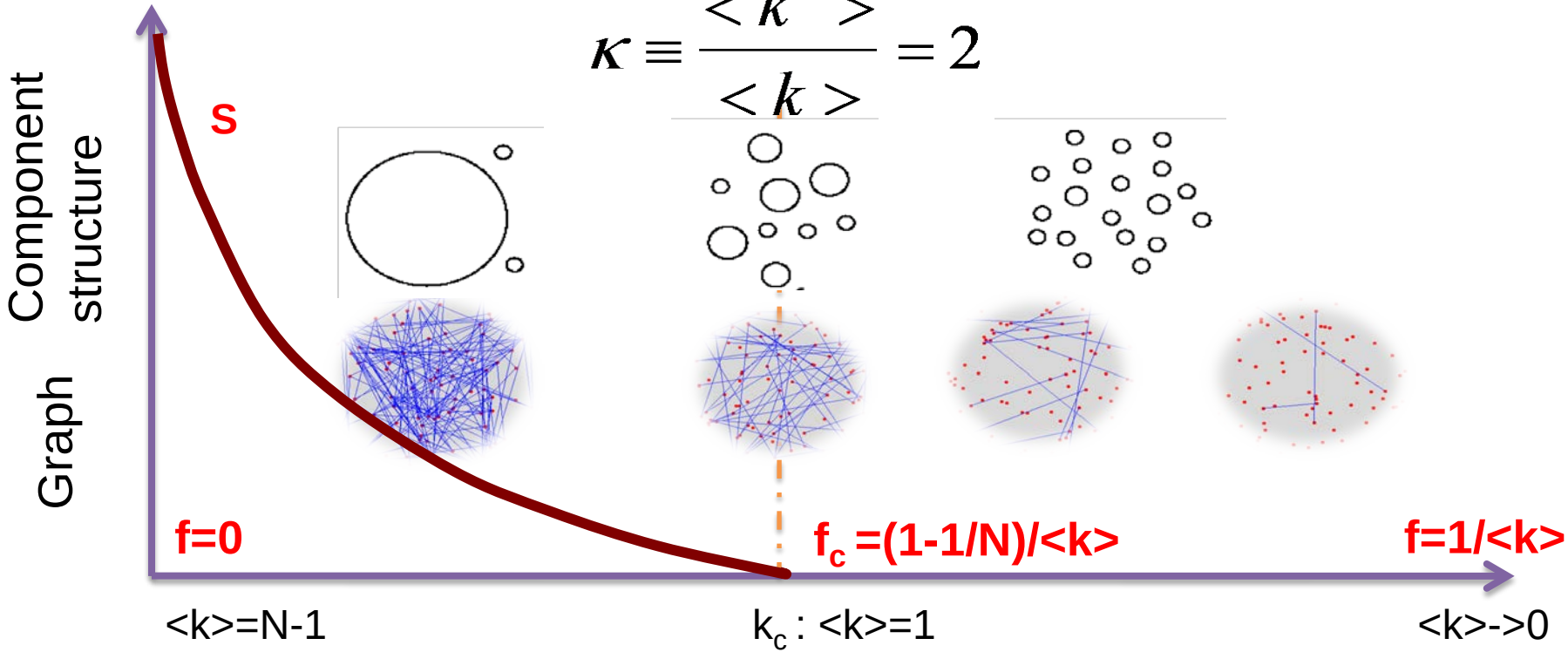
$$<k> = 1$$

Malloy-Reed; Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

RANDOM NETWORK: DAMAGE IS MODELED AS AN INVERSE OERCOLATION PROCESS

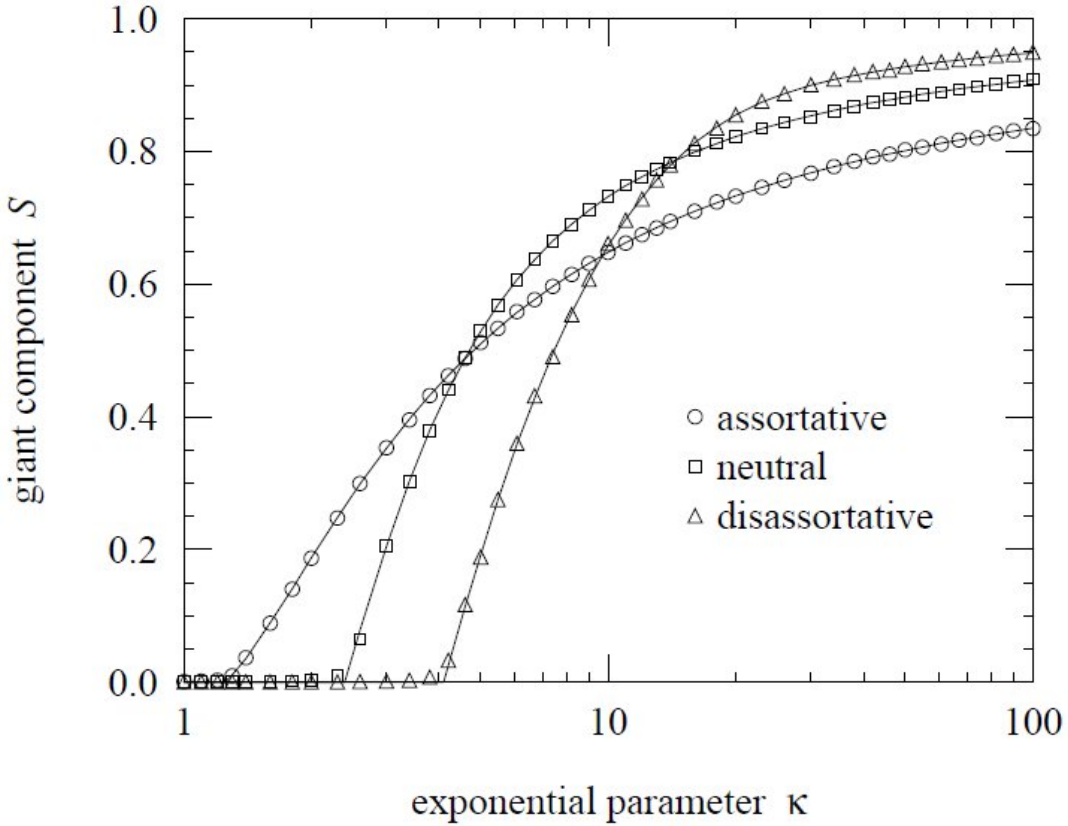
f = fraction of removed nodes

$$\kappa \equiv \frac{<k^2>}{<k>} = 2$$



(Inverse percolation phase transition)

FINAL REMARKS: EFFECT OF ASSORTATIVE MIXING: PERCOLATION



M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)

BREAKDOWN THRESHOLD FOR ARBITRARY P(k)

Problem: What are the consequences of removing a fraction *f* of all nodes?

At what threshold f_c will the network fall apart (no giant component)?

Random node removal changes

the degree of individual nodes [k → k' ≤k]

the degree distribution [P(k) → P'(k')]

A node with degree *k* will loose some links and become a node with degree *k'* with probability:

$$\binom{k}{k'} f^{k-k'}(1-f)^{k'} \qquad k' \le k$$

Remove k-k' links, each with probability f

Leave k' links untouched, each with probability 1-f

The prob. that we had a *k* degree node was *P(k)*, so the probability that we will have a new node with degree *k'* :

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'}(1-f)^{k'}$$

Let us asume that we know <k> and <k^2> for the original degree distribution P(k) → calculate <k'> , <k'^2> for the new degree distribution P'(k').

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

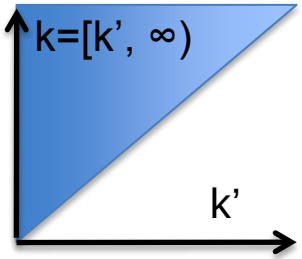
BREAKDOWN THRESHOLD FOR ARBITRARY P(K)

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'}$$

Degree distribution after we removed *f* fraction of nodes.

$$< k' >_f = \sum_{k'=0}^{\infty} k' P'(k') = \sum_{k'=0}^{\infty} k' \sum_{k=k'}^{\infty} P(k) \frac{k!}{k'!(k-k')!} f^{k-k'} (1-f)^{k'} = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} (1-f)$$

The sum is done over the triangle shown in the right, so we can replace it with



$$\sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} = \sum_{k=0}^{\infty} \sum_{k'=0}^{k}$$

$$< k' >_f = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} (1-f) = \sum_{k=0}^{\infty} (1-f) k P(k) \sum_{k'=0}^{k} \frac{(k-1)!}{(k'-1)!(k-k')!} f^{k-k'} (1-f)^{k'-1} =$$

$$\sum_{k=0}^{\infty} (1-f) k P(k) \sum_{k'=0}^{k} \binom{k-1}{k'-1} f^{k-k'} (1-f)^{k'-1} = \sum_{k=0}^{\infty} (1-f) k P(k) = \boxed{(1-f) < k >}$$

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

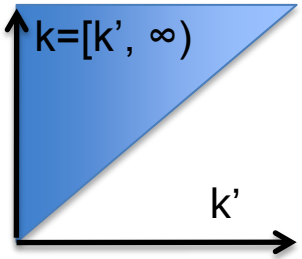
BREAKDOWN THRESHOLD FOR ARBITRARY P(K)

$$P'(k') = \sum_{k=k'}^{\infty} P(k) \binom{k}{k'} f^{k-k'} (1-f)^{k'}$$

Degree distribution after we removed f fraction of nodes.

$$< k'^2 >_f = < k'(k'-1) - k' >_f = \sum_{k'=0}^{\infty} k'(k'-1) P'(k') - < k' >_f$$

The sum is done over the triangle shown in the right, i.e. we can replace it with



$$\sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} = \sum_{k=0}^{\infty} \sum_{k'=0}^{k}$$

$$< k'(1-k') >_f = \sum_{k'=0}^{\infty} \sum_{k=k'}^{\infty} P(k) \frac{k(k-1)(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k-2'} (1-f)^2 = \sum_{k=0}^{\infty} (1-f)^2 k(k-1) P(k) \sum_{k'=0}^{k} \frac{(k-2)!}{(k'-2)!(k-k')!} f^{k-k'} (1-f)^{k'-2} =$$

$$\sum_{k=0}^{\infty} (1-f)^2 k(k-1) P(k) \sum_{k'=0}^{k} \binom{k-2}{k'-2} f^{k-k'} (1-f)^{k'-2} = \sum_{k=0}^{\infty} (1-f)^2 k(k-1) P(k) = (1-f)^2 < k(k-1) >$$

$$< k'^2 >_f = < k'(k'-1) - k' >_f = (1-f)^2 (< k^2 > - < k >) - (1-f) < k > = \underline{(1-f)^2 < k^2 > + f(1-f) < k >}$$

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

BREAKDOWN THRESHOLD FOR ARBITRARY P(K)

Robustness: we remove a fraction *f* of the nodes.

At what threshold f_c will the network fall apart (no giant component)?

Random node removal changes

the degree of individuals nodes [k → k' ≤k)

the degree distribution [P(k) → P'(k')]

$$< k' >_f = (1-f) < k >$$

$$< k'^2 >_f = (1-f)^2 < k^2 > + f(1-f) < k >$$

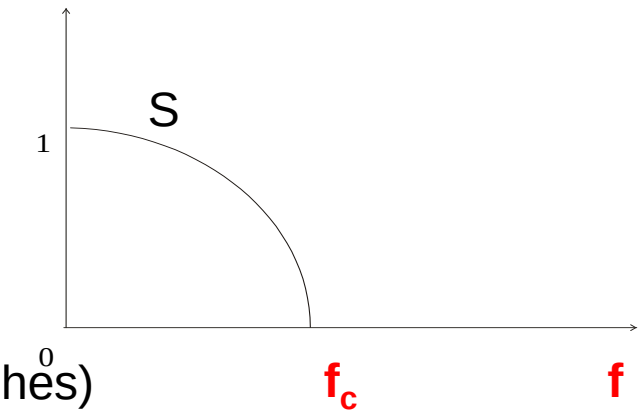
$$\kappa \equiv \frac{< k'^2 >_f}{< k' >_f} = 2$$

κ>2: a giant cluster exists

κ<2: many disconnected clusters

Breakdown threshold:

$$\boxed{f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}}$$



f<f_c: the network is still connected (there is a giant cluster)

f>f_c: the network becomes disconnected (giant cluster vanishes)

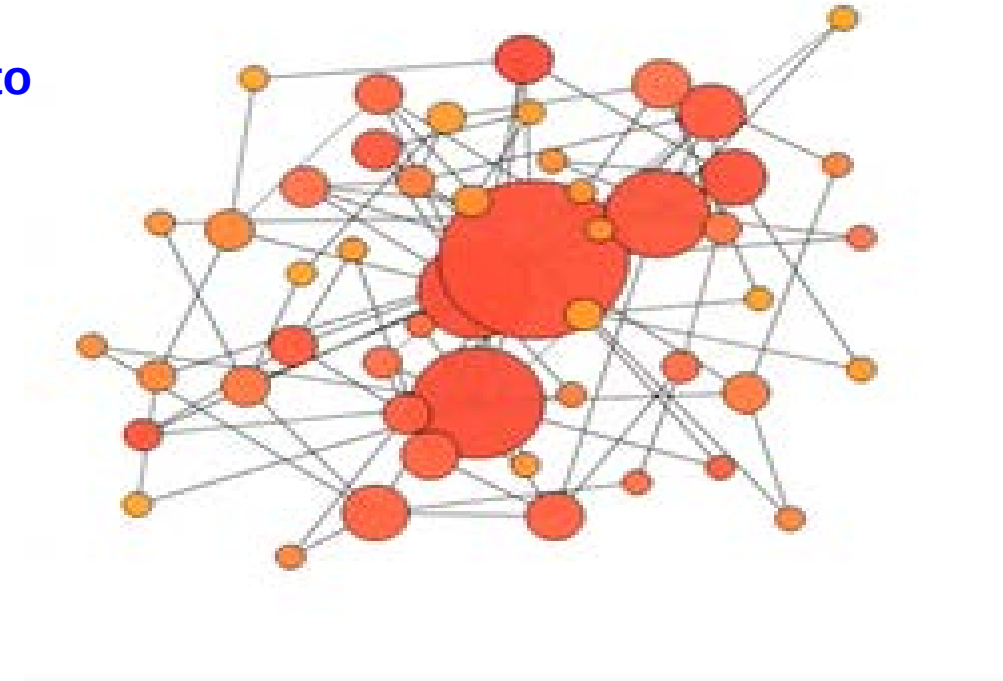
Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

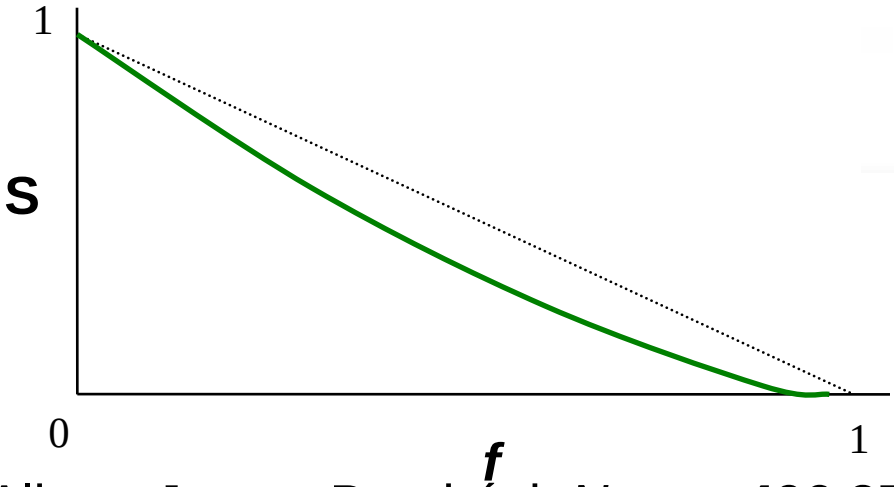
ROBUSTNESS OF SCALE-FREE NETWORKS

Scale-free networks do not appear to break apart under random failures.

Reason: the hubs.

The likelihood of removing a hub is small.





Albert, Jeong, Barabási, *Nature* **406** 378 (2000)

ROBUSTNESS OF SCALE-FREE NETWORKS

Scale-free networks do not appear to break apart under random failures. Why is that?

$$f_c = 1 - \frac{1}{\frac{\langle k^2 \rangle}{\langle k \rangle} - 1}$$

$$< k^m > = (\gamma - 1) K_{\min}^{\gamma-1} \int_{K_{\min}}^{K_{\max}} k^{m-\gamma} dk = \frac{(\gamma-1)}{(m-\gamma+1)} K_{\min}^{\gamma-1} \left[k^{m-\gamma+1} \right]_{K_{\min}}^{K_{\max}}$$

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

$$< k^m > = \frac{(\gamma-1)}{(m-\gamma+1)} K_{\min}^{\gamma-1} \left[K_{\max}{}^{m-\gamma+1} - K_{\min}{}^{m-\gamma+1} \right]$$

$$\frac{< k^2 >}{< k >} = \frac{(2-\gamma)}{(3-\gamma)} \frac{K_{\max}{}^{3-\gamma} - K_{\min}{}^{3-\gamma}}{K_{\max}{}^{2-\gamma} - K_{\min}{}^{2-\gamma}}$$

$$\gamma > 3: \quad \kappa = \frac{(2-\gamma)}{(3-\gamma)} \frac{K_{\max}{}^{3-\gamma} - K_{\min}{}^{3-\gamma}}{K_{\max}{}^{2-\gamma} - K_{\min}{}^{2-\gamma}} = \left| \frac{2-\gamma}{3-\gamma} \right| K_{\min}$$

$$3 > \gamma > 2: \quad \kappa = \frac{(2-\gamma)}{(3-\gamma)} \frac{K_{\max}{}^{3-\gamma} - K_{\min}{}^{3-\gamma}}{K_{\max}{}^{2-\gamma} - K_{\min}{}^{2-\gamma}} = \left| \frac{2-\gamma}{3-\gamma} \right| K_{\max}{}^{3-\gamma} K_{\min}^{\gamma-2}$$

$$2 > \gamma > 1: \quad \kappa = \frac{(2-\gamma)}{(3-\gamma)} \frac{K_{\max}{}^{3-\gamma} - K_{\min}{}^{3-\gamma}}{K_{\max}{}^{2-\gamma} - K_{\min}{}^{2-\gamma}} = \left| \frac{2-\gamma}{3-\gamma} \right| K_{\max}$$

ROBUSTNESS OF SCALE-FREE NETWORKS

$$f_c = 1 - \frac{1}{\kappa - 1} \qquad \kappa = \frac{<k^2>}{<k>} = \left|\frac{2-\gamma}{3-\gamma}\right| \begin{cases} K_{\min} & \gamma > 3 \\ K_{\max}{}^{3-\gamma} K_{\min}^{\gamma-2} & 3 > \gamma > 2 \\ K_{\max} & 2 > \gamma > 1 \end{cases}$$

$$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

γ>3: κ is finite, so the network will break apart at a finite f_c that depens on K_{min}

γ<3: κ diverges in the N→ ∞ limit, so f_c → 1 !!!
for an infinite system one needs to remove all the nodes to break the system.

For a finite system, there is a finite but large f_c that scales with the system size as: $\kappa \cong 1 - CN^{-\frac{3-\gamma}{\gamma-1}}$

Internet: Router level map, N=228,263; γ=2.1±0.1; κ=28 → f_c=0.962

AS level map, N= 11,164; γ=2.1±0.1; κ=264 → f_c=0.996

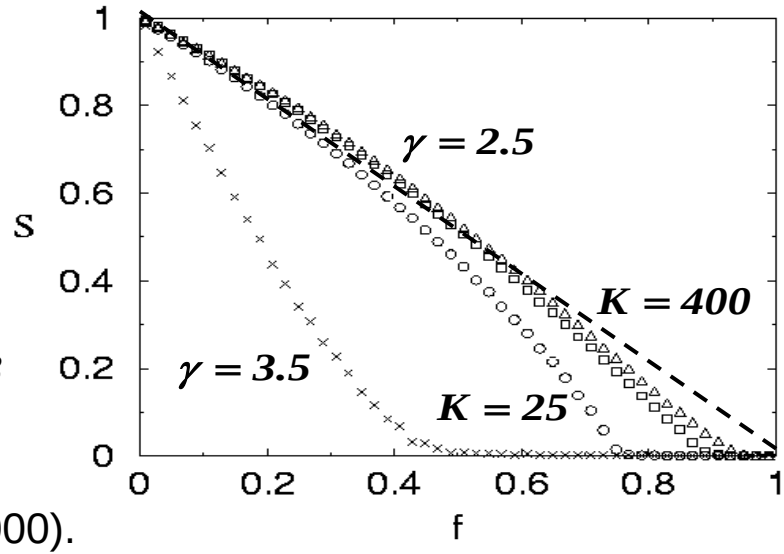
NUMERICAL EVIDENCE

Scale-free random graph with

$$P(k) = Ak^{-\gamma}, \text{ with } k = m,\ldots K$$

$$f_c = 1 - \frac{1}{\frac{\gamma-2}{\gamma-3}m - 1} \quad \text{if} \quad \gamma > 3$$

$$f_c = 1 - \frac{1}{\frac{\gamma-2}{3-\gamma}m^{\gamma-2}K^{3-\gamma} - 1} \quad \text{if} \quad 2 < \gamma < 3$$



Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Infinite scale-free networks with $\gamma < 3$ do not break down under random node failures.

SIZE OF THE GIANT COMPONENT DURING RANDOM DAMAGE –WITHOUT PROOF-

S: size of the giant component, f fraction of randomly removed nodes, not damage for $f<f_c$

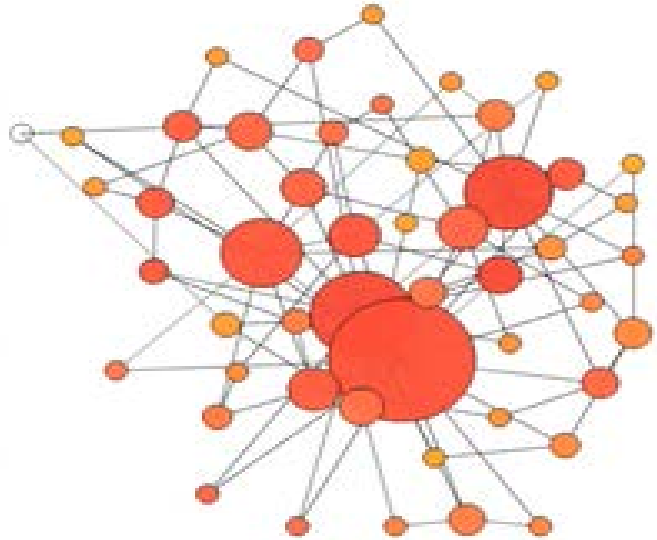
(i) $\gamma>4$: $S \approx f-f_c$ (similar to that of a random graph)

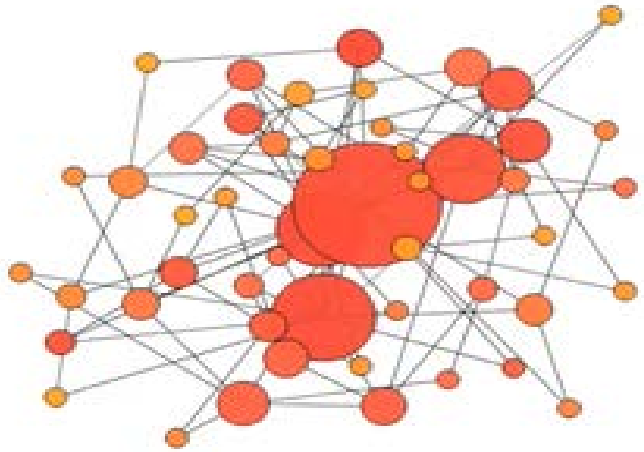
(i) $3>\gamma>4$: $S \approx (f-f_c)^{1/(\gamma-3)}$

(i) $\gamma<3$: $f_c=0$ and $S \approx f^{1+1/(3-\gamma)}$

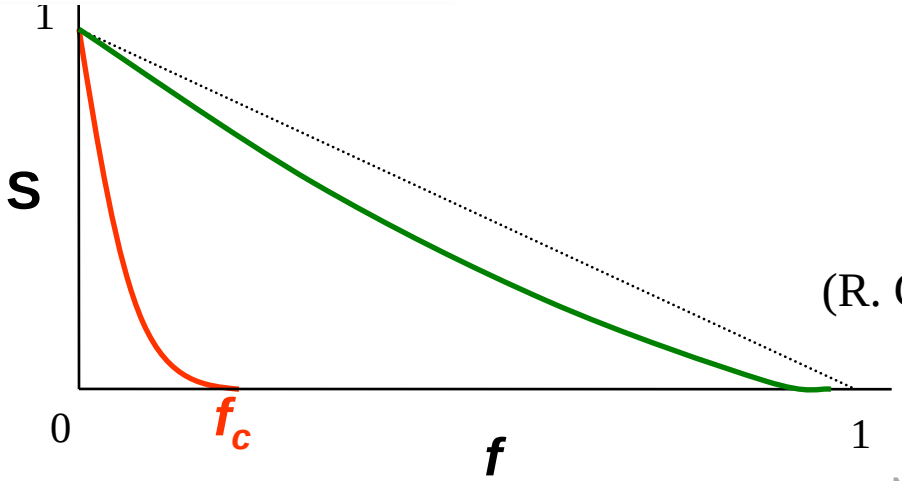
R. Cohen, D. ben-Avraham, S. Havlin,
Percolation critical exponents in scale-free networks
Phys. Rev. E 66, 036113 (2002);
See also: Dorogovtsev S, Lectures on Complex Networks, Oxford, pg44

Achilles' Heel of scale-free networks





Attacks



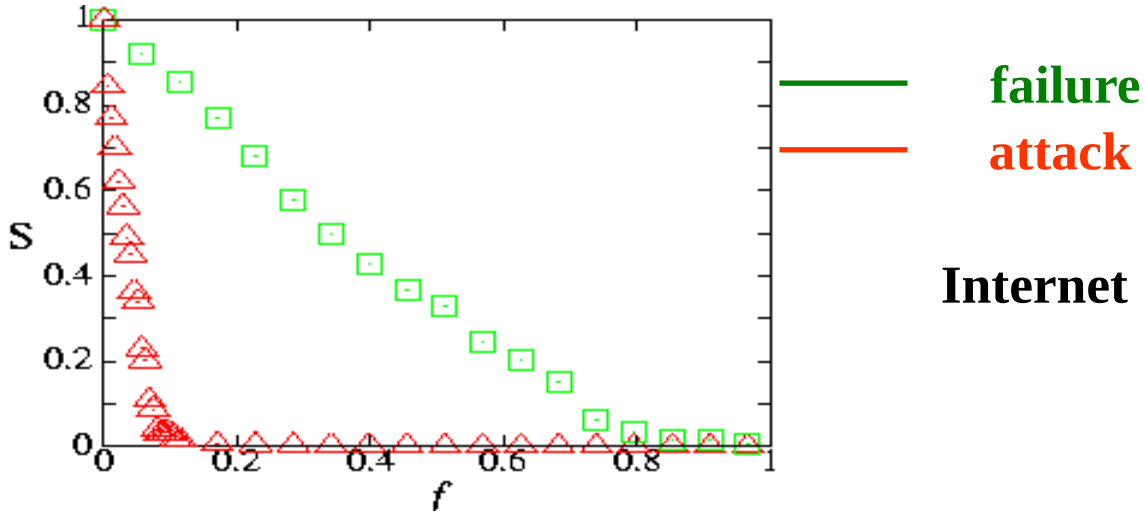
Failures

$\gamma \leq 3 : f_c = 1$

(R. Cohen et al PRL, 2000)

Albert, Jeong, Barabási, *Nature* **406** 378 (2000)

INTERNET'S ROBUSTNESS TO RANDOM FAILURES



R. Albert, H. Jeong, A.L. Barabasi, *Nature* **406** 378 (2000)

$$f_c = 1 - \frac{1}{\kappa - 1}$$

Internet: Router level map, N=228,263; γ=2.1±0.1; $\kappa=28$ → $f_c=0.962$

AS level map, N= 11,164; γ=2.1±0.1; $\kappa=264$ → $f_c=0.996$

Internet parameters: Pastor-Satorras & Vespignani, *Evolution and Structure of the Internet*: Table 4.1 & 4.4

Attack threshold for arbitrary P(k)

Attack problem: we remove a fraction *f* of the hubs.

At what threshold f_c will the network fall apart (no giant component)?

Hub removal changes

the maximum degree of the network [$K_{max} \rightarrow K'_{max} \leq K_{max}$)

the degree distribution [$P(k) \rightarrow P'(k')$]

A node with degree k will loose some links because some of its neighbors will vanish.

Claim: once we correct for the changes in K_{max} and P(k),we are back to the robustness problem.

That is, attack is nothing but a robusiness of the network with a new K_{max} and P(k).

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Attack threshold for arbitrary P(k)

Attack problem: we remove a fraction *f* of the hubs.

the maximum degree of the network [K_{max} → $K'_{max} \leq K_{max}$) `

If we remove an *f* fraction of hubs, the maximum degree changes:

$$\int_{K'_{max}}^{K_{max}} P(k)dk = f$$

$$\int_{K'_{max}}^{K_{max}} P(k)dk = (\gamma-1)K_{min}^{\gamma-1}\int_{K'_{max}}^{K_{max}} k^{-\gamma}dk = \frac{\gamma-1}{1-\gamma}K_{min}^{\gamma-1}(K_{max}^{1-\gamma} - K'^{1-\gamma}_{max})$$

As $K'_{max} \leq K_{max}$ we can ignore the K_{max} term

$$\left(\frac{K_{min}}{K'_{max}}\right)^{\gamma-1} = f \qquad K'_{max} = K_{min} f^{\frac{1}{1-\gamma}}$$

← The new maximum degree after removing f fraction of the hubs.

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Attack threshold for arbitrary P(k)

Attack problem: we remove a fraction *f* of the hubs.

the degree distribution changes [P(k) → P'(k')]

A node with degree *k* will loose some links because some of its neighbors will vanish.

Let us calculate the fraction of links removed 'randomly' , f', as a consequence of we removing f fraction of hubs.

$$\int_0^{K_{max}} kP(k)dk$$

as $K'_{max} \leq K_{max}$

$$f' = \frac{\int_{K'_{max}}^{K_{max}} kP(k)dk}{<k>} = \frac{1}{<k>}(\gamma-1)K_{min}^{\gamma-1}\int_{K'_{max}}^{K_{max}} k^{1-\gamma}dk = \frac{1}{<k>}\frac{\gamma-1}{2-\gamma}K_{min}^{\gamma-1}(K_{max}^{2-\gamma} - K'^{2-\gamma}_{max}) = -\frac{1}{<k>}\frac{\gamma-1}{2-\gamma}K_{min}^{\gamma-1}K'^{2-\gamma}_{max}$$

$$f' = -\frac{1}{<k>}\frac{\gamma-1}{2-\gamma}K_{min}^{\gamma-1}K_{min}^{2-\gamma}f^{\frac{2-\gamma}{1-\gamma}} = -\frac{1}{<k>}\frac{\gamma-1}{2-\gamma}K_{min}f^{\frac{2-\gamma}{1-\gamma}}$$

$$K'_{max} = K_{min}f^{\frac{1}{\gamma-1}}$$

$$<k^m> = -\frac{(\gamma-1)}{(m-\gamma+1)}K_{min}^m$$

$$f' = f^{\frac{2-\gamma}{1-\gamma}}$$

$$<k> = -\frac{(\gamma-1)}{(2-\gamma)}K_{min}$$

For γ→2, f'→1, which means that even the removal of a tiny fraction of hubs will destroy the network. The reason is that for γ=2 hubs dominate the network

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).