Frontiers of Network Science
Fall 2018

Class 16: Degree Correlations II
(Chapter 7 in Textbook)

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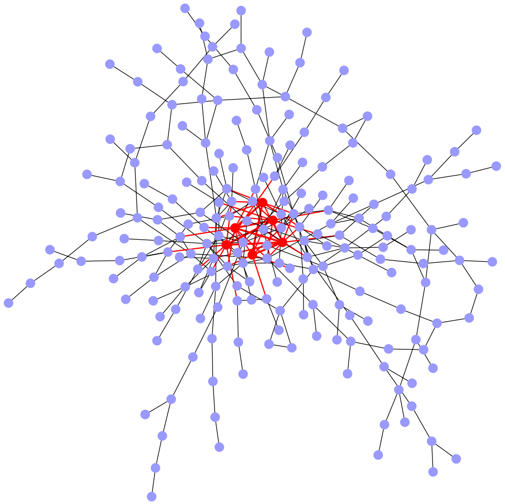
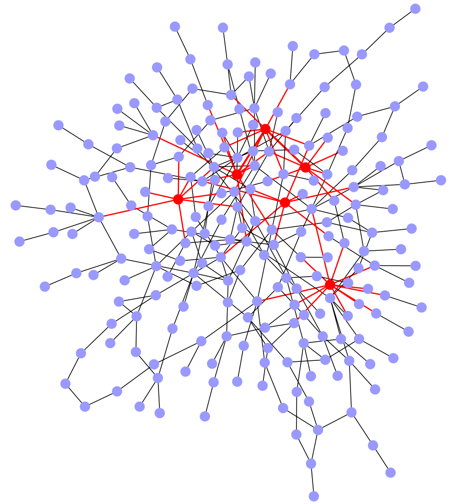
based on slides by
Albert-László Barabási
and Roberta Sinatra

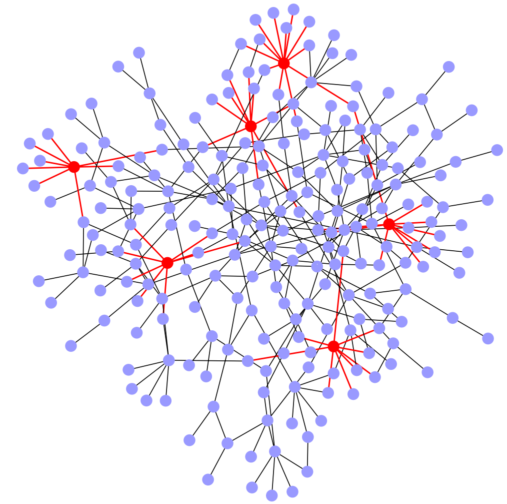
DEGREE CORRELATIONS IN NETWORKS



Assortative:
hubs show a tendency to link to each other.



Neutral:
nodes connect to each other with the expected random probabilities.



Disassortative:
Hubs tend to avoid linking to each other.

Quantifying degree correlations (three approaches):

→ full statistical description (Maslov and Sneppen, Science 2001)
→ degree correlation function (Pastor Satorras and Vespignani, PRL 2001)
→ correlation coefficient (Newman, PRL 2002)

STATISTICAL DESCRIPTION

e_{jk}: probability to find a node with degree *j* and degree *k* at the two ends of a randomly selected edge

$$\sum_{j,k} e_{jk} = 1 \qquad \sum_{j} e_{jk} = q_k$$

q_k: the probability to have a degree *k* node at the end of a link.

Where: $$q_k = \frac{kp_k}{\langle k \rangle}$$

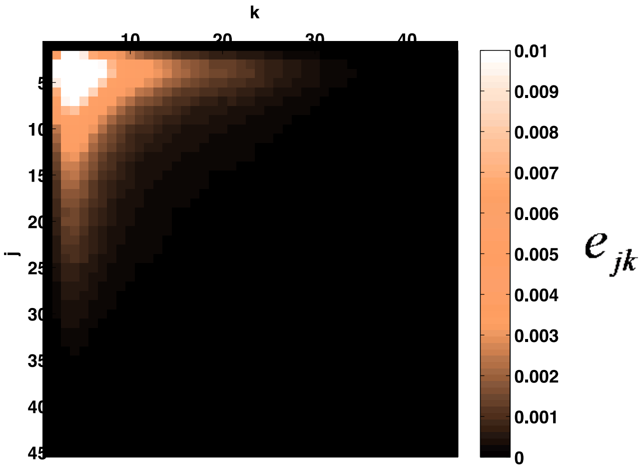
Probability to find a node at the end of a link is biased towards the more connected nodes, i.e. $q_k = Ckp_k$, where C is a normalization constant . After normalization we find $C = 1/\langle k \rangle$, or $q_k = kp_k/\langle k \rangle$

If the network has no degree correlations: $$e_{jk} = q_j q_k$$

Deviations from this prediction are a signature of *degree correlation.*

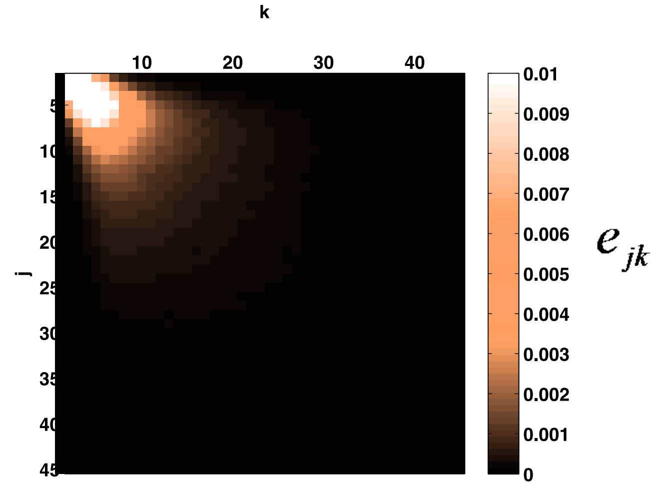
M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)

EXAMPLE: e_{jk} FOR A SCALE-FREE NETWORK

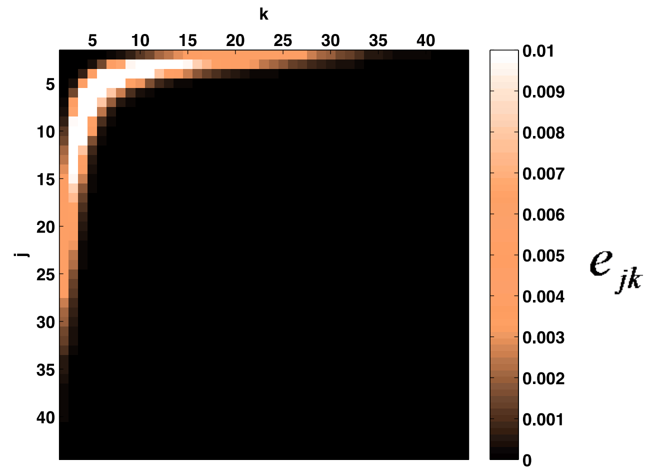


Neutral

Assortative: More strength in the diagonal, hubs tend to link to each other.



Disassortative: Hubs tend to connect to small nodes.



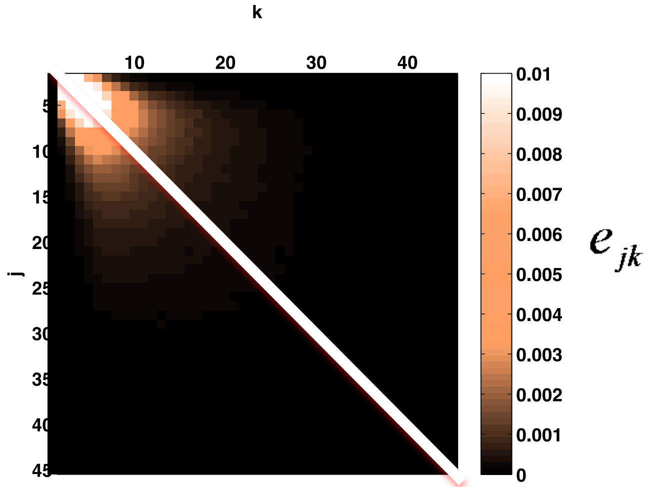
Each matrix is the average of a 100 independent scale-free networks, generated using the static model with $N=10^4$, $\gamma=2.5$ and $\langle k \rangle=3$.

EXAMPLE: e_{jk} FOR A SCALE-FREE NETWORK

Perfectly assortative network:

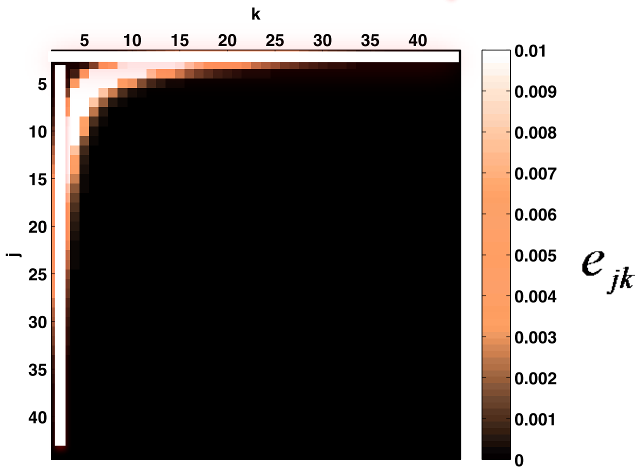
$e_{jk}=q_k\delta_{jk}$

Assortative:
More strength in the diagonal, hubs tend to link to each other.



Perfectly disassortative network:

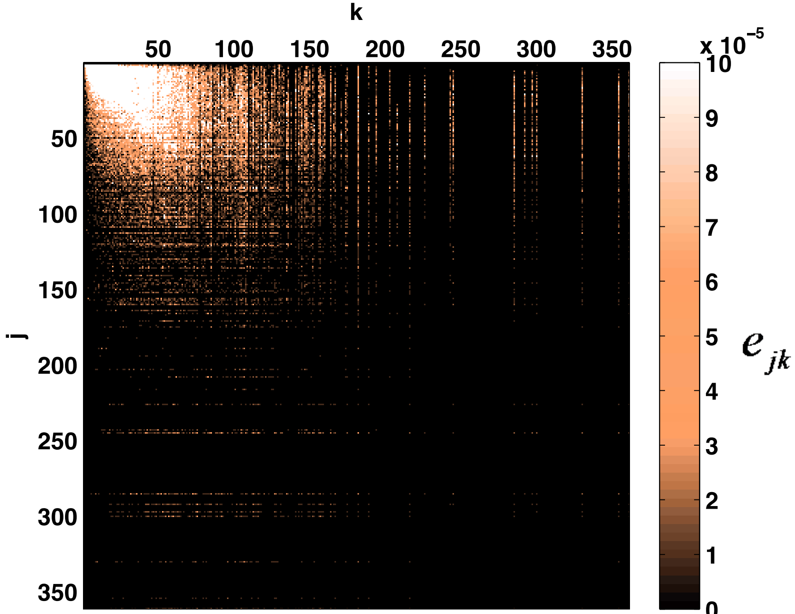
Disassortative:
Hubs tend to connect to small nodes.



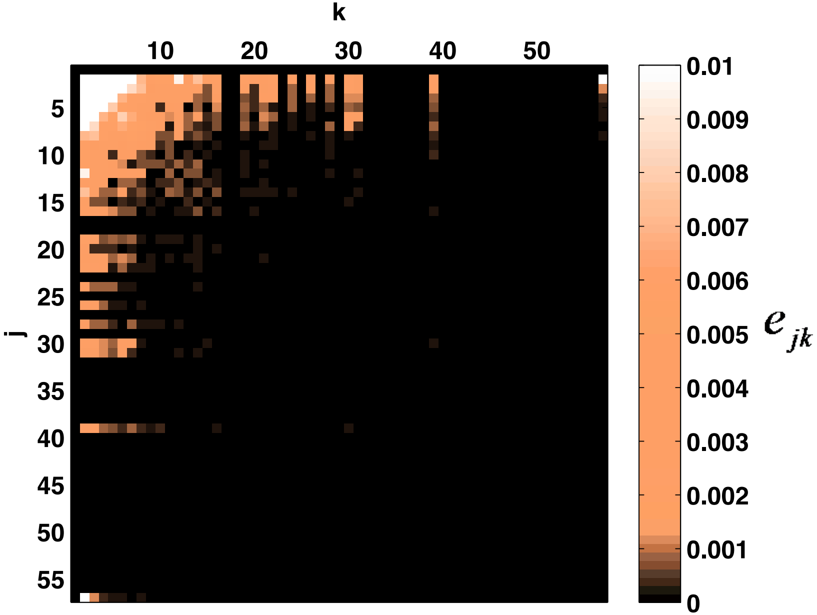
Each matrix is the average of a 100 independent scale-free networks, generated using the static model with $N=10^4$, $\gamma=2.5$ and $\langle k \rangle=3$.

REAL-WORLD EXAMPLES

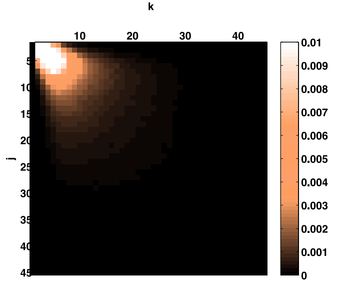
Astrophysics co-authorship network



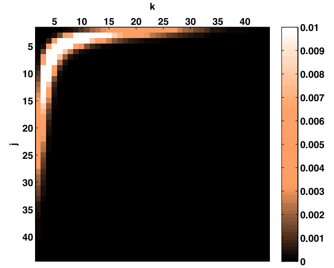
Yeast PPI



Assortative: More strength in the diagonal, hubs tend to link to each other.

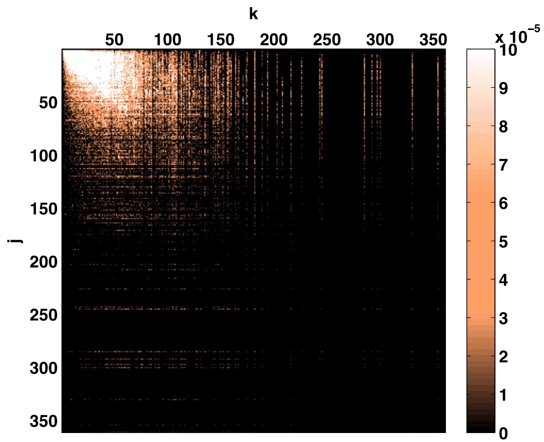


Disassortative: Hubs tend to connect to small nodes.



PROBLEM WITH THE FULL STATISTICAL DESCRIPTION

(1) Difficult to extract information from a visual inspection of a matrix.



(2) Based on e_{jk} and hence requires a large number of elements to inspect:

$$\frac{k_{\max}\left(k_{\max}-1\right)}{2}-1-k_{\max}$$

Nr. of independent elements

Undirected network:
k_{max} x k_{max} matrix

$$\sum_{j,k} e_{jk} = 1 \qquad \sum_{j=1,k_{\max}} e_{jk} = q_k$$

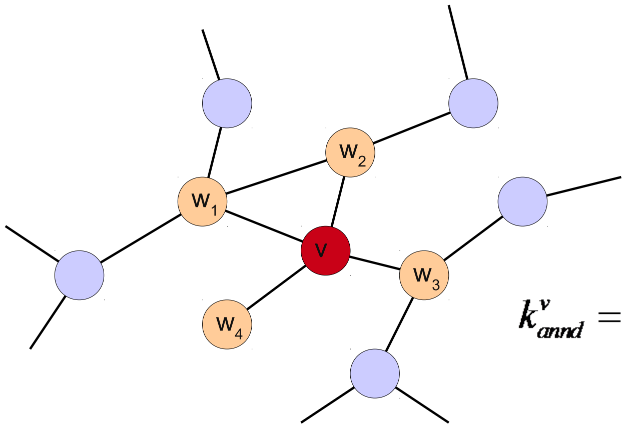
Constraints

We need to find a way to reduce the information contained in e_{jk}

M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)

Average next neighbor degree

$k_{annd}(k)$: average degree of the first neighbors of nodes with degree k.



$$k^{v}_{annd} = \frac{4+3+3+1}{4}$$

$$k_{annd}(k) = \sum_{k'} k' P(k' \mid k) = \frac{\sum_{k'} k' e_{kk'}}{\sum_{k'} e_{kk'}}$$

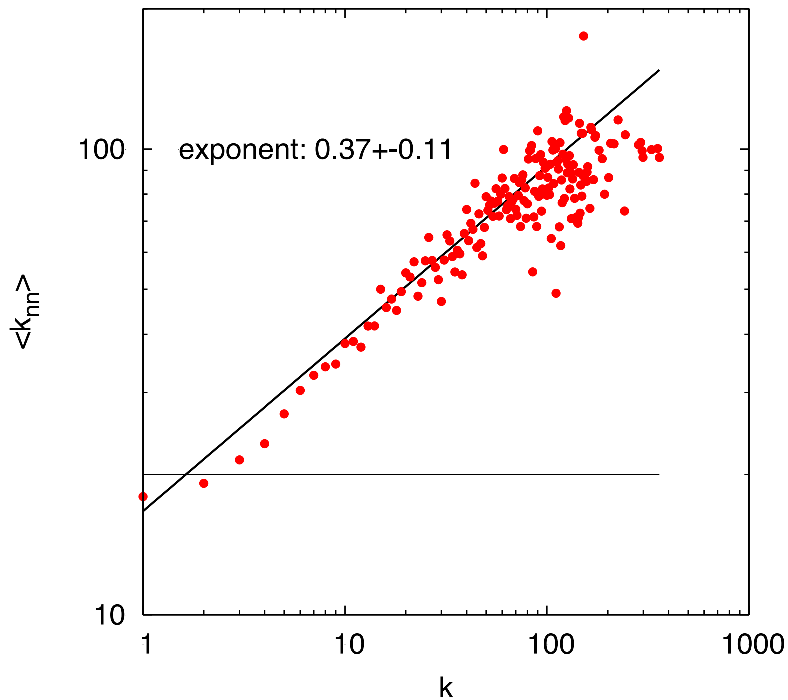
No degree correlations:

$$k_{annd}(k) = \frac{\sum_{k'} k' e_{kk'}}{\sum_{k'} e_{kk'}} = \frac{\sum_{k'} k' q_k q_{k'}}{q_k} = \sum_{k'} k' q_{k'} = \sum_{k'} k' \frac{k' p(k')}{< k >} = \frac{\langle k^2 \rangle}{\langle k \rangle}$$

If there are no degree correlations, $k_{annd}(k)$ is independent of k.

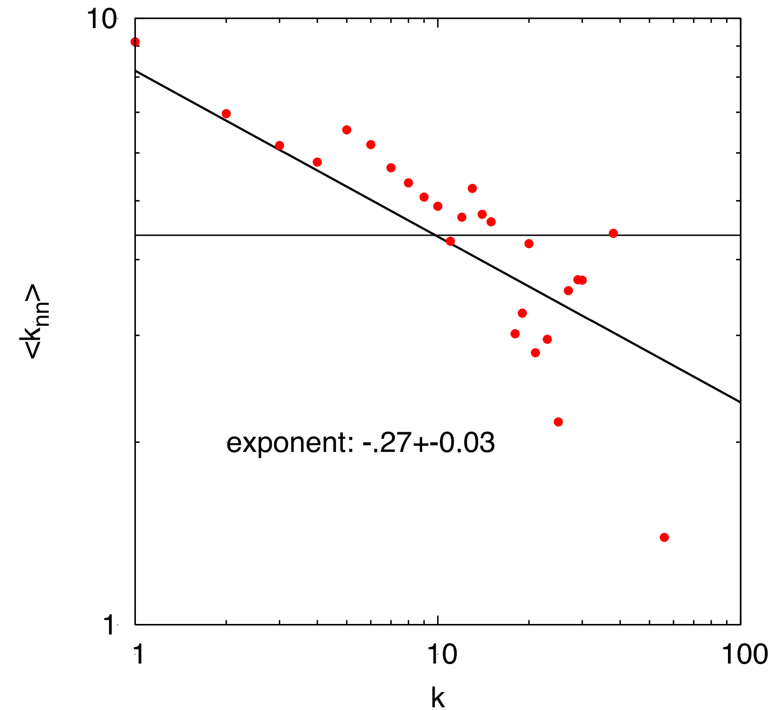
R. Pastor-Satorras, A. Vázquez, A. Vespignani, Phys. Rev. E **65**, 066130 (2001)

$k_{annd}(k)$ FOR REAL NETWORKS



Astrophysics co-authorship network

Assortative



Yeast PPI

Disassortative

Average next neighbor degree

$k_{annd}(k)$: average degree of the first neighbors of nodes with degree k.

constraint:

$$\sum_k k_{annd}(k) \cdot kNp_k = \sum_k k^2 \cdot Np_k$$

$$\langle k_{annd}(k)k \rangle = \langle k^2 \rangle$$

$\longrightarrow$ k_{max}-1 independent elements

$k_{annd}(k)$ is a k-dependent function, hence it has much fewer parameters,

and it is easier to interpret/read.

R. Pastor-Satorras, A. Vázquez, A. Vespignani, Phys. Rev. E **65**, 066130 (2001)

PEARSON CORRELATION

If there are degree correlations, e_{jk} will differ from $q_j q_k$. The magnitude of the correlation is captured by <jk>-<j><k> difference, which is:

$$\sum_{jk} jk(e_{jk} - q_j q_k)$$

<jk>-<j><k> is expected to be:

positive for *assortative* networks,
zero for *neutral* networks,
negative for *dissasortative* networks

To compare different networks, we should normalize it with its maximum value; the maximum is reached for a *perfectly assortative network*, i.e. $e_{jk}=q_k\delta_{jk}$

normalization: $$\sigma_r^2 = \max \sum_{jk} jk(e_{jk} - q_j q_k) = \sum_{jk} jk(q_k \delta_{jk} - q_j q_k)$$

$$r = \frac{\sum_{jk} jk(e_{jk} - q_j q_k)}{\sigma_r^2} \qquad -1 \le r \le 1$$

$r \le 0$ *disassortative*
$r = 0$ *neutral*
$r \ge 0$ *assortative*

M. E. J. Newman, Phys. Rev. Lett. **89,** 208701 (2002)

REAL NETWORKS

Network	n	r
Physics coauthorship (a)	52 909	0.363
Biology coauthorship (a)	1 520 251	0.127
Mathematics coauthorship (b)	253 339	0.120
Film actor collaborations (c)	449 913	0.208
Company directors (d)	7 673	0.276
Internet (e)	10 697	−0.189
World-Wide Web (f)	269 504	−0.065
Protein interactions (g)	2 115	−0.156
Neural network (h)	307	−0.163
Marine food web (i)	134	−0.247
Freshwater food web (j)	92	−0.276
Random graph (u)		0
Callaway *et al.* (v)		$\delta/(1+2\delta)$
Barabási and Albert (w)		0

Social networks are *assortative*

Biological, technological networks are *disassortative*

r>0: assortative network:
Hubs tend to connect to other hubs.

r<0: disassortative network:
Hubs tend to connect to small nodes.

RELATIONSHIP BETWEEN r AND k_{annd}

$$r = \frac{\sum_{kj} kj(e_{kj} - q_k q_j)}{\sigma_r^2} = \frac{\sum_k k q_k \sum_j \frac{j e_{kj}}{q_k} - \left(\sum_k k q_k\right)^2}{\sigma_r^2} = \frac{\sum_k k k_{annd}(k) q_k - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2}$$

$$k_{annd}(k) = \sum_{k'} k' P(k' \mid k) = \frac{\sum_{k'} k' e_{kk'}}{\sum_{k'} e_{kk'}} = \frac{\sum_{k'} k' e_{kk'}}{q_k}$$

In general case we need to know q_k and $k_{annd}(k)$ to calculate r.

Assuming: $k_{annd}(k) = a \cdot k + b$

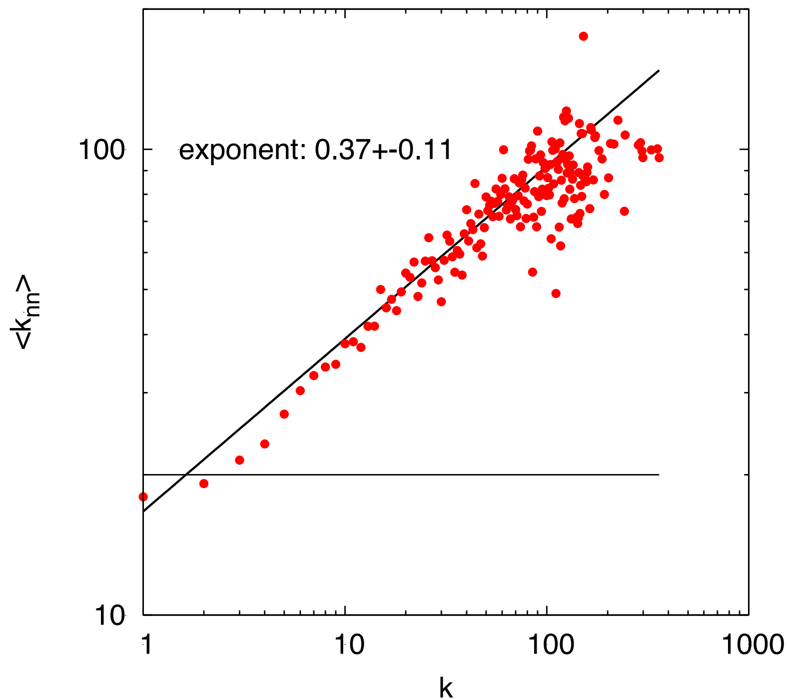
Using the constraint for ANND:

$$\langle k^2 \rangle = \langle k_{annd}(k) k \rangle = \sum_{k'} a \cdot k^2 p_k + b \cdot k p_k = a\langle k^2 \rangle + b\langle k \rangle \longrightarrow b = \frac{(1-a)\langle k^2 \rangle}{\langle k \rangle}$$

$$r = \frac{\sum_k k \cdot (a \cdot k + b) q_k - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} = \frac{\sum_k k \cdot \left(a \cdot k + \frac{(1-a)\langle k^2 \rangle}{\langle k \rangle}\right) \frac{k p_k}{\langle k \rangle} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} =$$

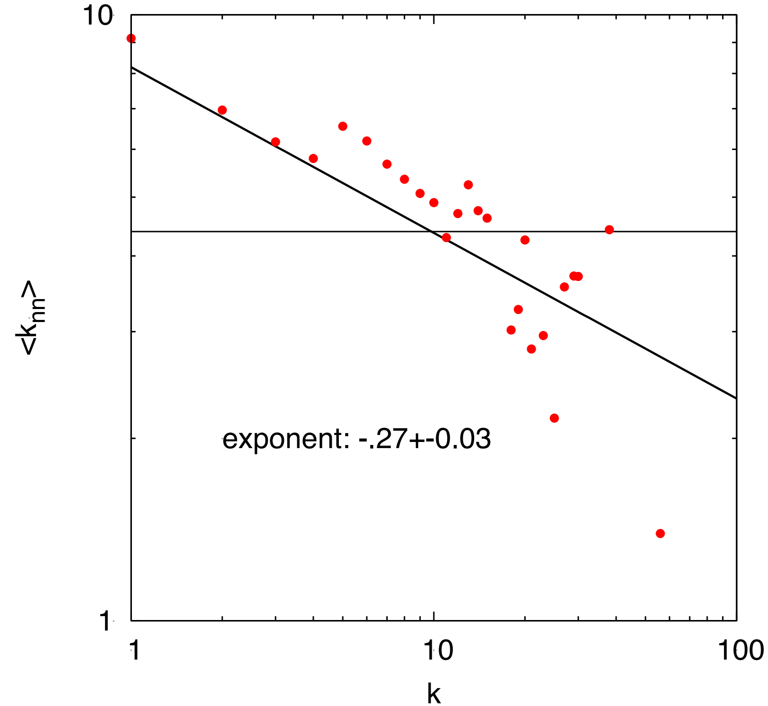
$$= \frac{a\left(\sum_k k^3 \frac{p_k}{\langle k \rangle} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}\right) + \frac{\langle k^2 \rangle^2}{\langle k \rangle^2} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} = a$$

PROBLEM WITH THE PREVIOUS DEVIATION: $k_{annd}(k)$~k^{β}



Astrophysics co-authorship network

Assortative



Yeast PPI

Disassortative

CONNECTION WITH *ANND*

Assuming: $k_{annd}(k) = a \cdot k^{\beta}$

Using the constraint for ANND: $\langle k^2 \rangle = \langle k_{annd}(k) k \rangle = \sum_{k'} a \cdot k^{\beta+1} p_k = a \langle k^{\beta+1} \rangle \longrightarrow a = \frac{\langle k^2 \rangle}{\langle k^{\beta+1} \rangle}$

$$r = \frac{\sum_k k \cdot a k^{\beta} \cdot q_k - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} = \frac{\sum_k a \cdot k^{\beta+2} \frac{p_k}{\langle k \rangle} \cdot - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} = \frac{\frac{\langle k^2 \rangle}{\langle k^{\beta+1} \rangle} \frac{\langle k^{\beta+2} \rangle}{\langle k \rangle} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}}{\sigma_r^2} =$$

$$= \frac{1}{\sigma_r^2} \frac{\langle k^2 \rangle}{\langle k \rangle} \left(\frac{\langle k^{\beta+2} \rangle}{\langle k^{\beta+1} \rangle} - \frac{\langle k^2 \rangle}{\langle k \rangle} \right)$$

$$\sigma_r^2 = \sum_{jk} jk(q_k \delta_{jk} - q_j q_k) = \frac{\langle k^3 \rangle}{\langle k \rangle} - \frac{\langle k^2 \rangle^2}{\langle k \rangle^2}$$

$$\beta < 0 \quad \rightarrow \quad r < 0$$

$$\beta = 0 \quad \rightarrow \quad r = 0$$

$$\beta > 0 \quad \rightarrow \quad r > 0$$

CONNECTION BETWEEN R AND k_{ANND}

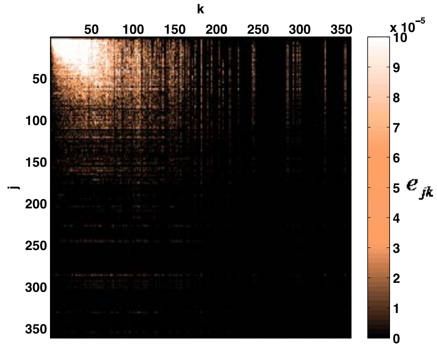
$$\beta = 0: \qquad \frac{\langle k^{\beta+2}\rangle}{\langle k^{\beta+1}\rangle} - \frac{\langle k^2\rangle}{\langle k\rangle} = \frac{\langle k^2\rangle}{\langle k\rangle} - \frac{\langle k^2\rangle}{\langle k\rangle} = 0 \quad \Rightarrow \quad r = 0$$

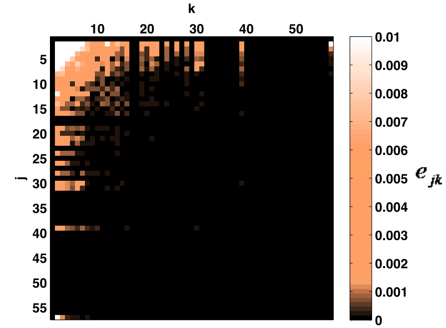
$$\langle k^{\alpha+\beta}\rangle = \sum_{k_{\min}}^{k_{\max}} k^{\alpha+\beta} p_k \begin{cases} < k_{\max}^{\beta} \sum_{k_{\min}}^{k_{\max}} k^{\alpha} p_k = k_{\max}^{\beta}\langle k^{\alpha}\rangle \\ > k_{\min}^{\beta} \sum_{k_{\min}}^{k_{\max}} k^{\alpha} p_k = k_{\min}^{\beta}\langle k^{\alpha}\rangle \end{cases}$$

$$\longrightarrow \quad \begin{array}{l} 0 > \beta > -1: \quad \dfrac{\langle k^2\rangle}{\langle k\rangle} > \left(\dfrac{k_{\min}}{k_{\max}}\right)^{-\beta} \dfrac{\langle k^{\beta+2}\rangle}{\langle k^{\beta+1}\rangle} > \dfrac{\langle k^{\beta+2}\rangle}{\langle k^{\beta+1}\rangle} \Rightarrow r < 0 \\ +1 > \beta > 0: \quad \dfrac{\langle k^{\beta+2}\rangle}{\langle k^{\beta+1}\rangle} > \left(\dfrac{k_{\min}}{k_{\max}}\right)^{\beta} \dfrac{\langle k^2\rangle}{\langle k\rangle} > \dfrac{\langle k^2\rangle}{\langle k\rangle} \Rightarrow r > 0 \end{array}$$

DEGREE CORRELATION IN NETWORKS

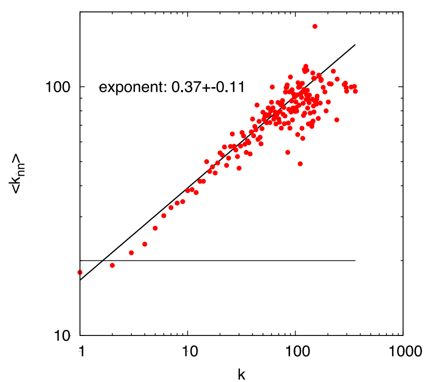
e_{jk}

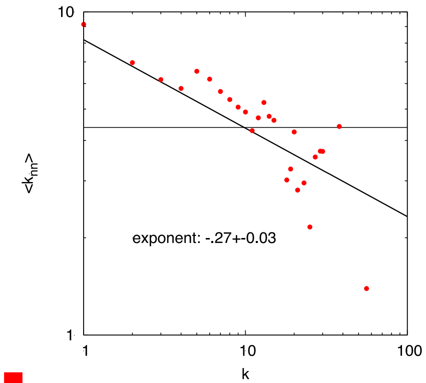




$$\frac{k_{\max}(k_{\max}-1)}{2}-k_{\max}-1$$

$k_{annd}(k)$





$k_{\max}-1$

r

0.31

-0.16

1

GENERATING NETWORK WITH GIVEN ASSORTATIVITY

We have a desired e_{jk} distribution, which also specifies p_k.

1. Generate a network with the desired degree distribution using the configuration model.
2. Choose two links at random from the network: (v_1,w_1) and (v_2,w_2).
3. Measure the degrees j_1, k_1, j_2, k_2 of nodes v_1, w_1, v_2, w_2. Replace the two selected links with two new ones (v_1,v_2) and (w_1,w_2) with probability

$$P = \begin{cases} \dfrac{e_{j_1 j_2} e_{k_1 k_2}}{e_{j_1 k_1} e_{k_2 j_2}} & \text{if } e_{j_1 j_2} e_{k_1 k_2} < e_{j_1 k_1} e_{k_2 j_2} \\ 1 & \text{otherwise} \end{cases}$$

1. Repeat from step 2.

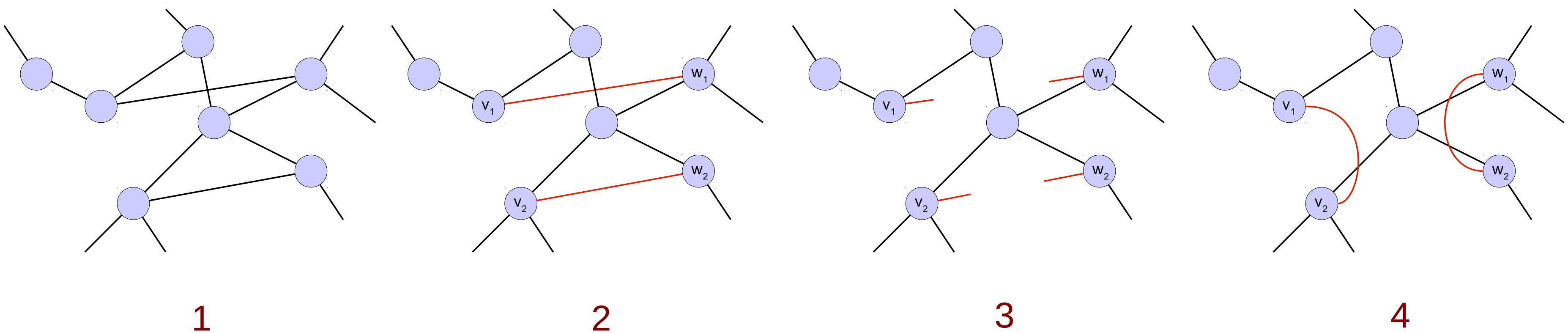
The algorithm is ergodic and satisfies detailed balance, therefore in the long time limit it samples the desired network ensemble correctly.

M. E. J. Newman, Phys. Rev. E **67,** 026126 (2003)

GENERATING NETWORK WITH GIVEN ASSORTATIVITY

2. Choose two edges random from the network: (v_1,w_1) and (v_2,w_2).
3. Measure the degrees j_1, k_1, j_2, k_2 of vertices v_1, w_1, v_2, w_2. Replace the two selected edges with two new ones (v_1,v_2) and (w_1,w_2) with probability

$$P = \begin{cases} \dfrac{e_{j_1 j_2} e_{k_1 k_2}}{e_{j_1 k_1} e_{k_2 j_2}} & \text{if } e_{j_1 j_2} e_{k_1 k_2} < e_{j_1 k_1} e_{k_2 j_2} \\ 1 & \text{otherwise} \end{cases}$$



GENERATING NETWORK WITH GIVEN ASSORTATIVITY

If we only specify *r* we have great degree of freedom in choosing e_{jk}.

Possible choice for disassortative case:

$$e_{jk}^{(d)} = q_j x_k + x_j q_k - x_j x_k$$

Where x_k is any normalized distribution.

This form satisfies the constraints on e_{jk}:

$$\sum_{jk} e_{jk} = \sum_{jk} q_k x_j + x_k q_j - x_k x_j = 1 + 1 - 1 = 1 \qquad \sum_{j} e_{jk} = \sum_{j} q_k x_j + x_k q_j - x_k x_j = q_k + x_k - x_k = q_k$$

The *r* value can be easily calculated:

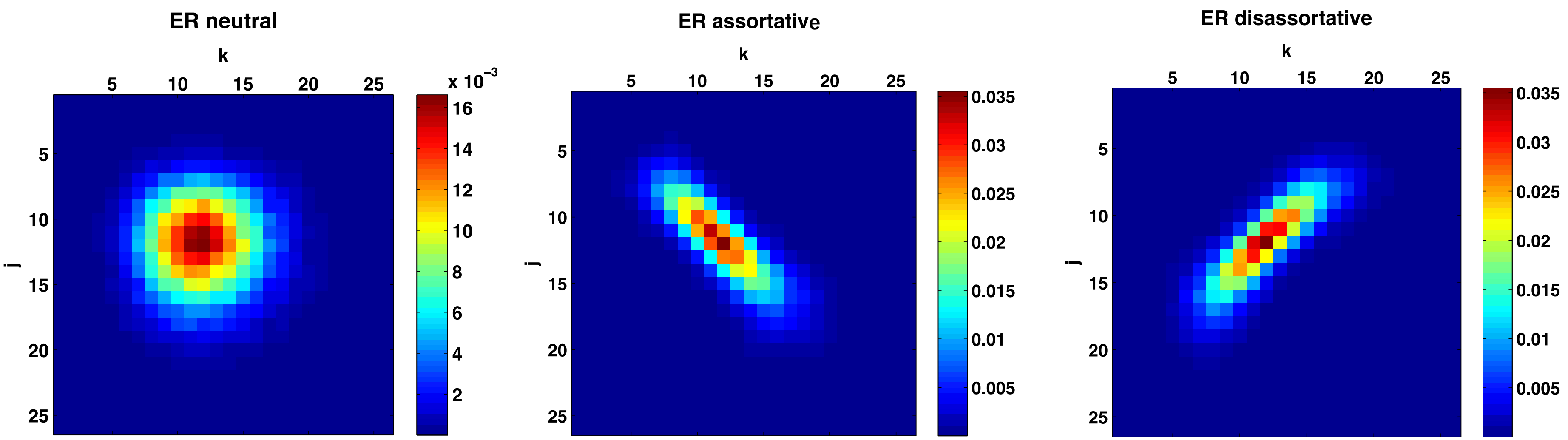
$$r_d = \frac{\sum_{jk} jk\left(q_k x_j + x_k q_j - x_k x_j - q_k q_j\right)}{\sigma_r^2} = \frac{2\langle k\rangle_q \langle k\rangle_x - \langle k\rangle_x^2 - \langle k\rangle_q^2}{\sigma_r^2} = -\frac{\left(\langle k\rangle_x - \langle k\rangle_q\right)^2}{\sigma_r^2}$$

Assortative case:

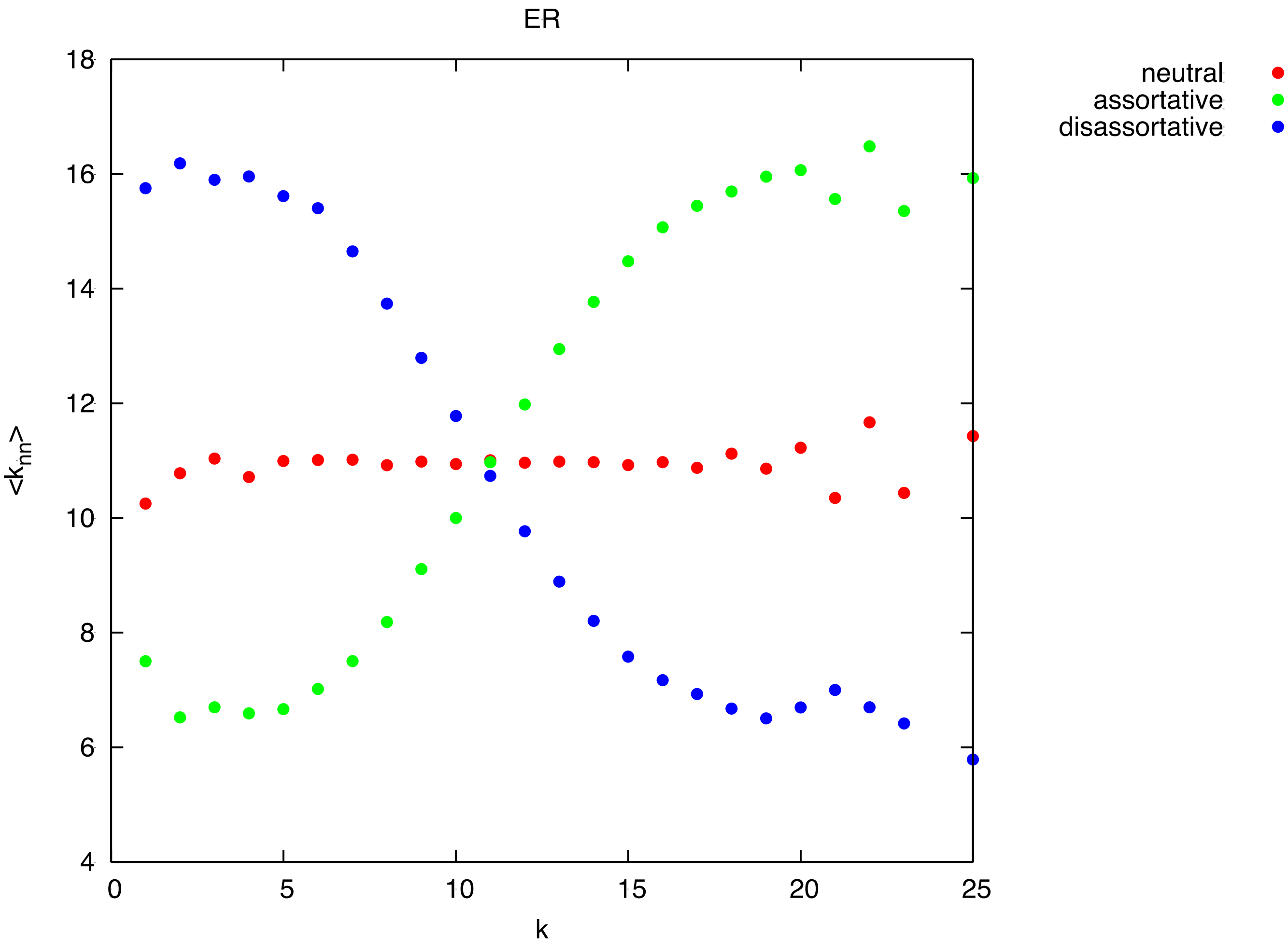
$$e_{jk}^{(a)} = q_j q_k - e_{jk}^{(d)} \longrightarrow r_a = -r_d$$

M. E. J. Newman, Phys. Rev. E **67,** 026126 (2003)

EXAMPLE: Erdős-Rényi



EXAMPLE: Erdős-Rényi



Structural cut-off

High assortativity→ high number of links between the hubs.

If we allow only one link between two nodes, we can simply run out of hubs to connect to each other to satisfy the assortativity criteria.

Number of edges between the set of nodes with degree *k* and degree *k'*:

$$E_{kk'} = e_{kk'} \langle k \rangle N$$

Maximum number of edges between the two groups:

$$m_{kk'} = \min\{kN_k, k'N_{k'}, N_k N_{k'}\}$$

There cannot be more links between the two groups, than the overall number of edges joining the nodes with degree k.

If we only have **simple edges**, we cannot have more links between the two groups, than if we connect every node with degree k to every node with degree k' **once**.

This is true even if we allow multiple edges.

M. Boguñá, R. Pastor-Satorras, A. Vespignani, EPJ B **38,** 205 (2004)

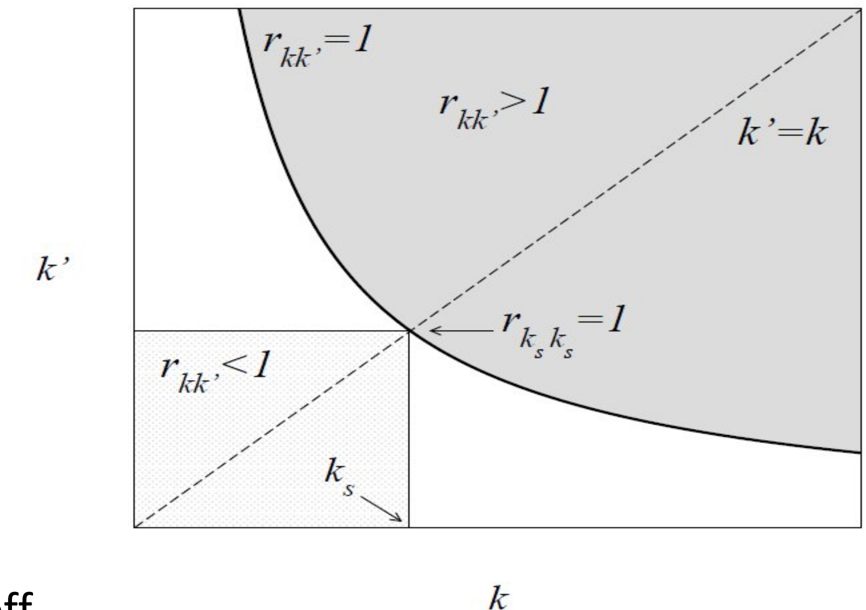
Structural cut-off

$$E_{kk'} = e_{kk'} \langle k \rangle N$$

$$m_{kk'} = \min\{kN_k, k'N_{k'}, N_k N_{k'}\}$$

The ratio of $E_{kk'}$ and $m_{kk'}$ has to be ≤ 1 in the physical region!

$$r_{kk'} = \frac{E_{kk'}}{m_{kk'}} \leq 1$$



$\longrightarrow$ $r_{k_s k_s} = 1$ defines the structural cut-off

M. Boguñá, R. Pastor-Satorras, A. Vespignani, EPJ B **38,** 205 (2004)

Structural cut-off for uncorrelated networks

Uncorrelated networks:

$$m_{kk'} = \min\{kN_k, k'N_{k'}, N_k N_{k'}\}$$

$$m_{k_s k_s} = k_s N_{k_s} = k_s N p_{k_s}$$

$$m_{k_s k_s} = N_{k_s}^2 = N^2 p_{k_s}^2$$

$$e_{kk'} = q_k q_{k'} = \frac{kk' p_k p_{k'}}{\langle k \rangle^2} \longrightarrow r_{kk'} = \frac{E_{kk'}}{m_{kk'}} = \frac{\langle k \rangle N e_{kk'}}{m_{kk'}}$$

$$r_{k_s k_s} = \frac{\langle k \rangle N \cdot k_s^2 \cdot p_{k_s}^2}{\langle k \rangle^2 k_s p_{k_s} N} = \frac{k_s p_{k_s}}{\langle k \rangle} = q_{k_s} < 1 \quad \forall k_s$$

$$r_{k_s k_s} = \frac{\langle k \rangle N \cdot k_s^2 \cdot p_{k_s}^2}{\langle k \rangle^2 N^2 \cdot p_{k_s}^2} = \frac{k_s^2}{\langle k \rangle N} \longrightarrow \boxed{k_s(N) = (\langle k \rangle N)^{1/2}}$$

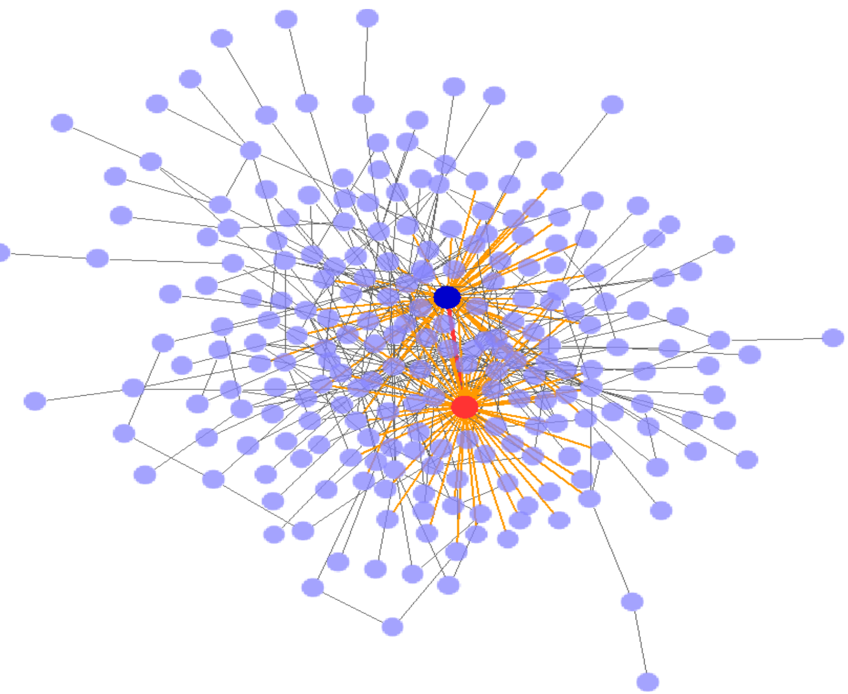
***$k_s(N)$* represents a *structural cutoff*:**
one cannot have nodes with degree larger than *$k_s(N)$* ,

→*if there are nodes with $k > k_s(N)$ we cannot find sufficient links between the highly connected nodes to maintain the neutral nature of the network.*

Solution:

(a) Introduce a structural cutoff (i.e. do not allow nodes with $k > k_s(N)$
(b) Let the network become more dissasortative, having fewer links between hubs.

Example: Degree sequence introduces disassortativity



Scale-free network generated with the configuration model (N=300, L=450, γ=2.2).

The measured r=-0.19! → Dissasortative!

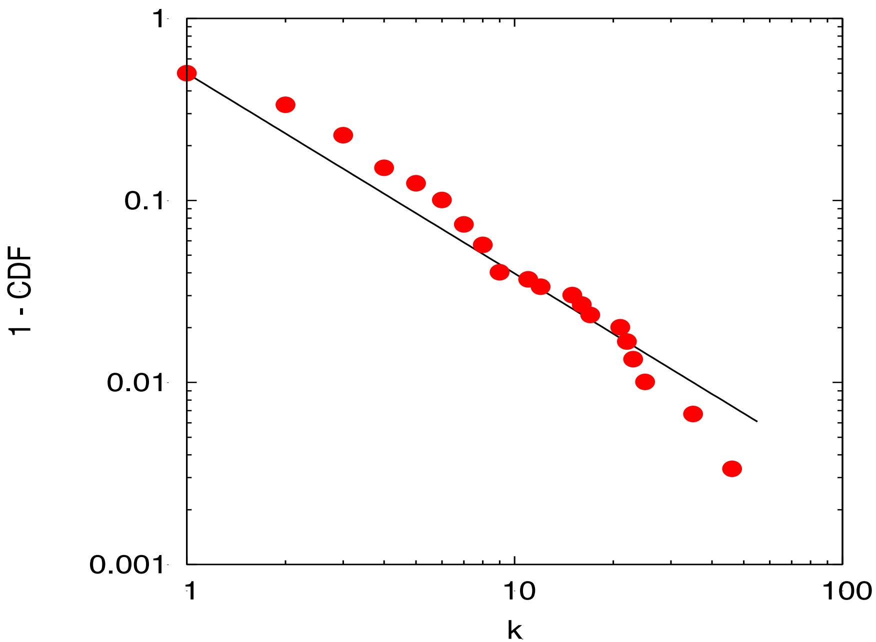
Red hub: 55 neighbors.
Blue hub: 46 neighbors.

Let's calculate the expectation number of links between red node (k=55) and blue node (k=46) for uncorrelated networks!

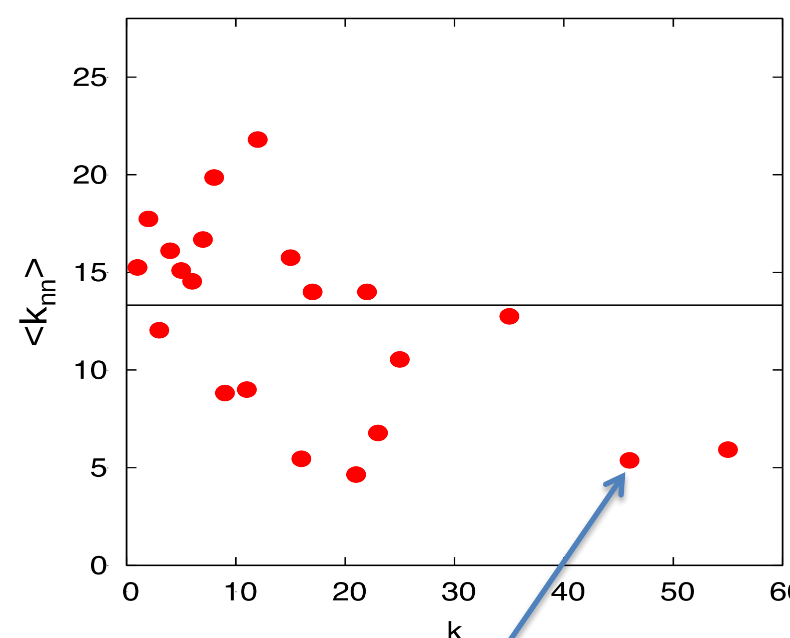
Here $N_{55}=N_{46}=1$, hence $m_{55,46}=1$ so $r_{55,46}=E_{55,46}$

$$E_{55,46}=\langle k\rangle N\cdot e_{55,46}=900\cdot\frac{\overset{k}{55}\,\overset{p_k}{\frac{1}{300}}\cdot\overset{k'}{46}\,\overset{p_{k'}}{\frac{1}{300}}}{\underset{\langle k\rangle}{3^2}}\approx 2.8>1$$

In order for the network to be neutral, we need 2.8 links between these two hubs.

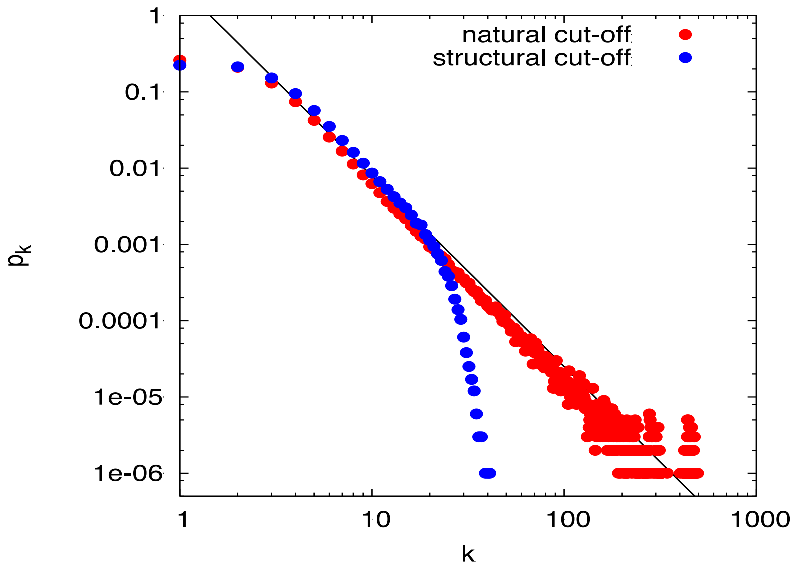


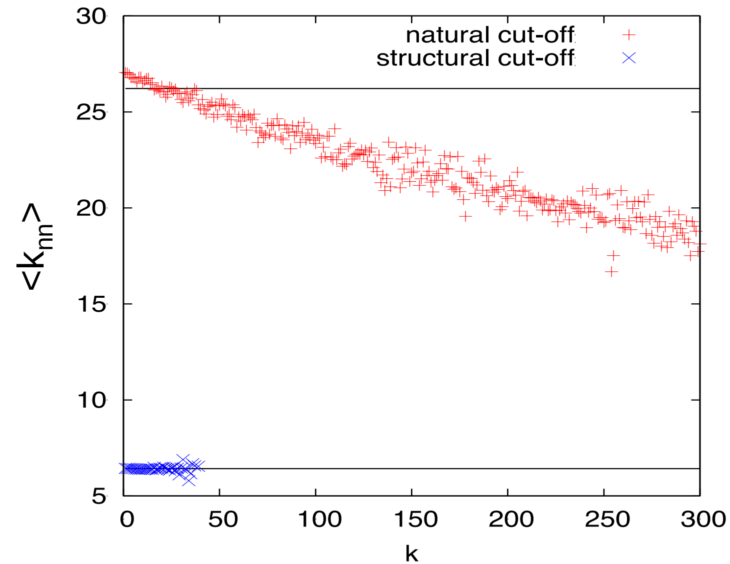
$$1 - CDF = P(k' > k) = 1 - \sum_{k'}^{k} p_{k'}$$



The largest nodes have $k_{nn} < <k_{nn}>$

The effect is particularly clear for N=10,000:





The red curves are those of interest to us: one can see that a clear dissasortativity property is visible in this case.

Natural cutoffs in scale-free networks

All real networks are finite → let us explore its consequences.
→ We have an expected maximum degree, K_{max}

<u>Estimating K_{max}</u>

$$\int_{K_{\max}}^{\infty} P(k)dk \approx \frac{1}{N}$$

Why: the probability to have a node larger than K_{max} should not exceed the prob. to have one node, i.e. *1/N* fraction of all nodes

$$\int_{K_{\max}}^{\infty} P(k)dk = (\gamma-1)K_{\min}^{\gamma-1}\int_{K_{\max}}^{\infty} k^{-\gamma}dk = \frac{(\gamma-1)}{(-\gamma+1)}K_{\min}^{\gamma-1}\left[k^{-\gamma+1}\right]_{K_{\max}}^{\infty} = \frac{K_{\min}^{\gamma-1}}{K_{\max}^{\gamma-1}} \approx \frac{1}{N}$$

Natural cutoff: $$K_{\max} = K_{\min} N^{\frac{1}{\gamma-1}}$$

Structural cut-off for uncorrelated networks

Structural cutoff: $\boxed{k_s(N) \sim (\langle k \rangle N)^{1/2}}$

$$e_{kk'} = q_k q_{k'} = \frac{kk' p_k p_{k'}}{\langle k \rangle^2}$$

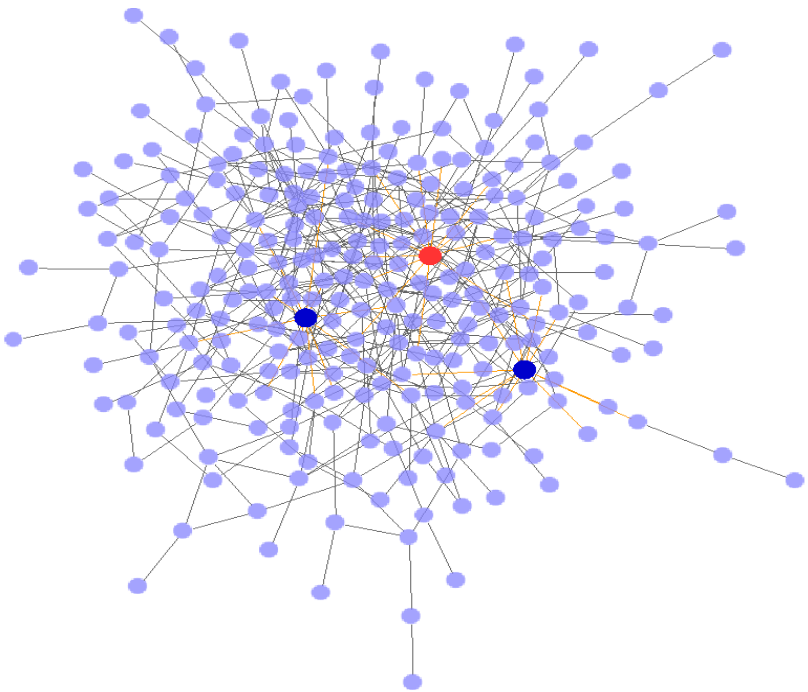
Natural cut-off: $k_{\max}(N) \sim N^{\frac{1}{\gamma-1}}$

γ=3: $k_s(N)$ and $k_{max}(N)$ scale the same way, i.e. $\sim N^{1/2}$.

γ<3: $k_{\max} > k_s$ $\longrightarrow$ The size of the largest hub is above the structural cutoff, which means that it cannot have enough links to the other hubs to maintain its neutral status.
→ disassortative mixing

→a randomly wired network with γ<3 will be
(a) dissasortative
(b) Or will have to have a cutoff at $k_s(N) < k_{max}(N)$

Example: introducing a structural cut-off



Scale-free network generated with the configuration model ($N=300$, $L=450$, $\gamma=2.2$) with structural cut-off ~ $N^{½}$.

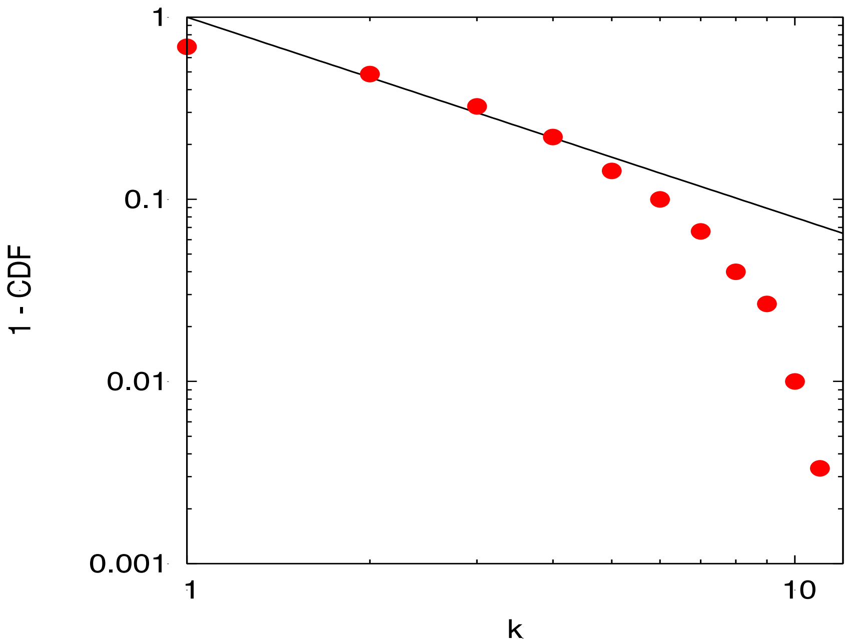
$r=0.005$ → neutral

Red hub: 12 neighbors.

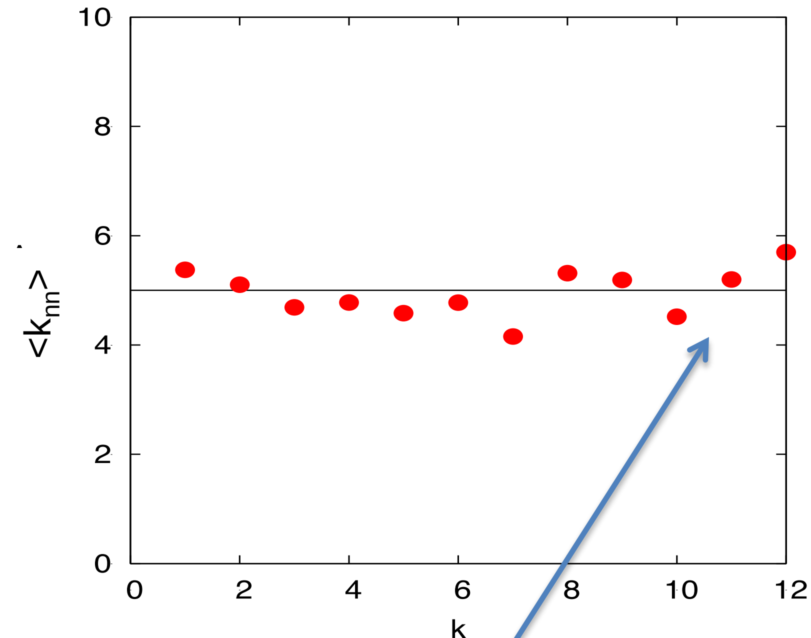
Blue hubs: 11 neighbors.

Again we can calculate the expectation number of edges between the hubs.

$$E_{11,12} = \langle k \rangle N \cdot e_{11,12} = 900 \cdot \frac{\overset{k}{12}\,\overset{p_k}{\frac{1}{300}} \cdot \overset{k'}{11}\,\overset{p_{k'}}{\frac{2}{300}}}{\underset{\langle k \rangle}{3^2}} \approx 0.3 < 1$$

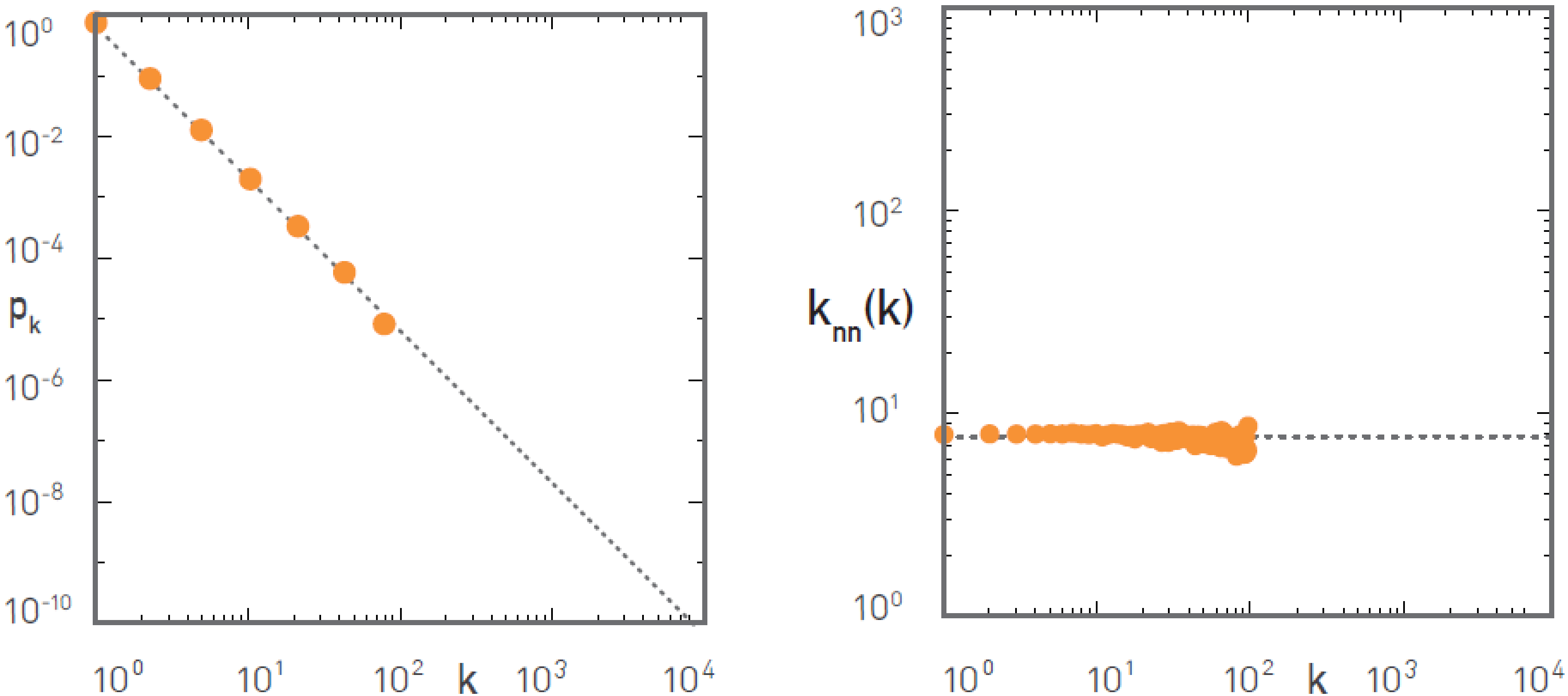


$$1 - CDF = P(k' > k) = 1 - \sum_{k'}^{k} p_{k'}$$



The largest nodes have k_{nn}~ $<k_{nn}>$

The effect is particularly clear for N=10,000:

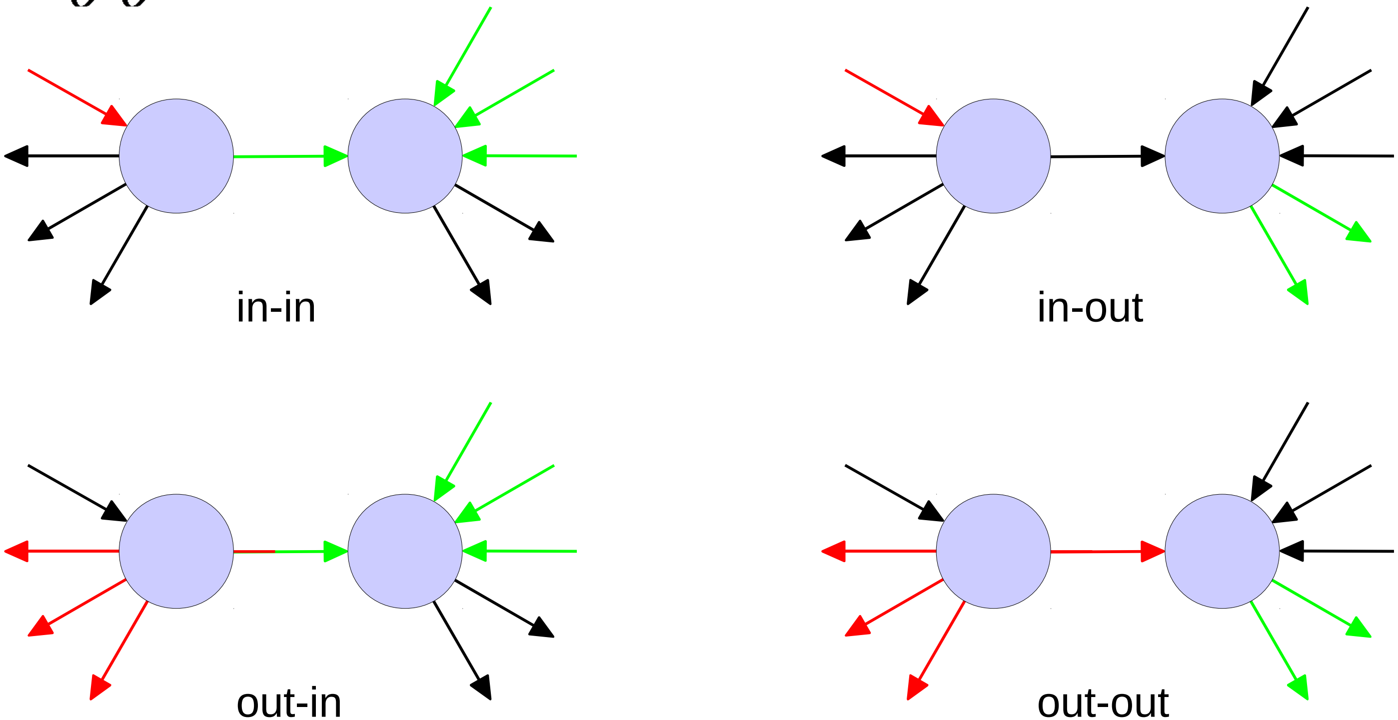


A clear case of neutral assortativity property is visible in this case thanks to imposing structural cut-off.

DIRECTED NETWORKS

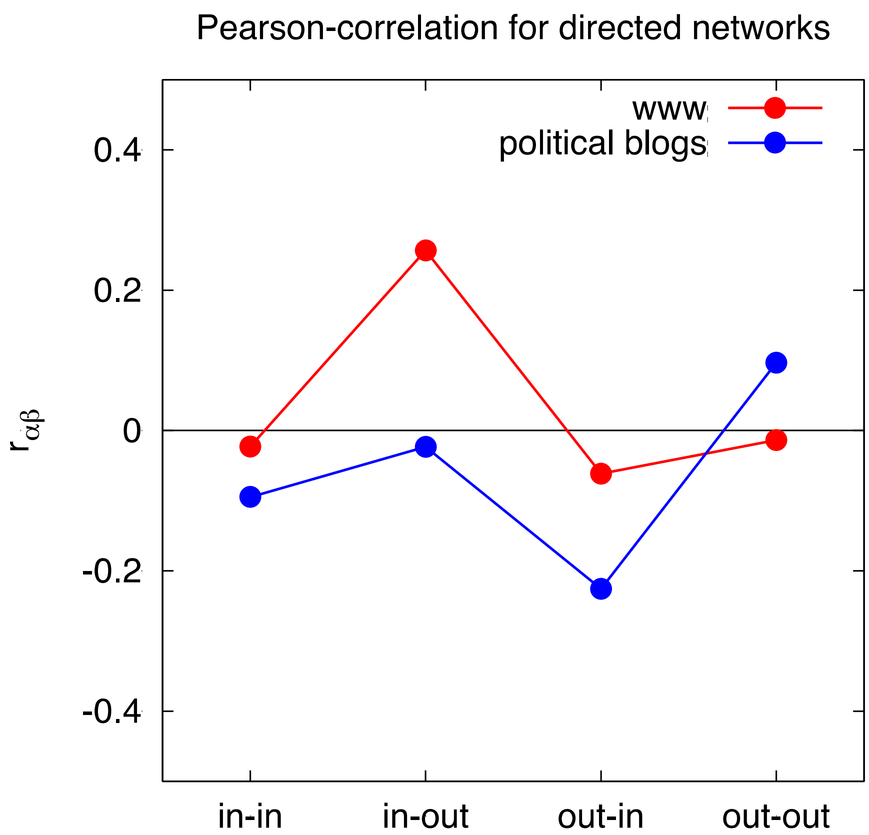
$$r_{\alpha\beta} = \frac{\sum_{jk} jk\left(e_{jk}^{\alpha\beta} - q_j^{\alpha} q_k^{\beta}\right)}{\sigma^{\alpha}\sigma^{\beta}}$$

α,β: {in,out}



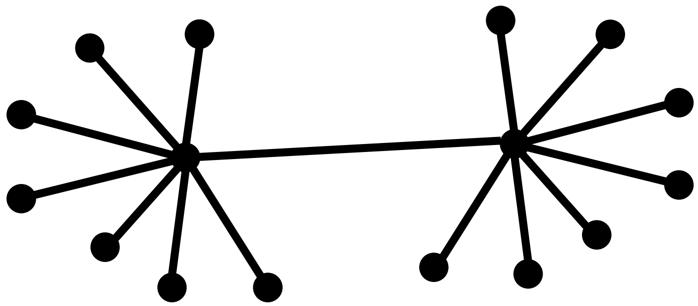
J. G. Foster, D. V. Foster, P. Grassberger, M. Paczuski, PNAS **107,** 10815 (2010)

DIRECTED NETWORKS

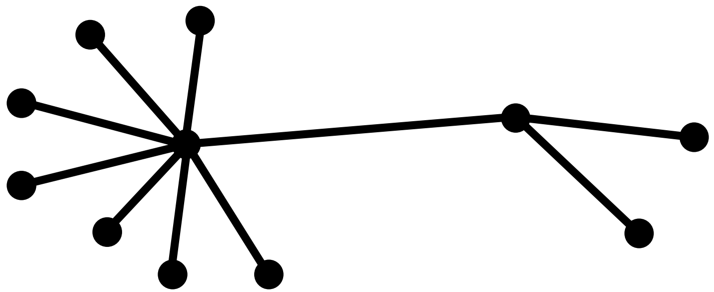


MULTIPOINT DEGREE CORRELATIONS

P(k): not enough to characterize a network



Large degree nodes tend to connect to large degree nodes
Ex: social networks



Large degree nodes tend to connect to small degree nodes
Ex: technological networks

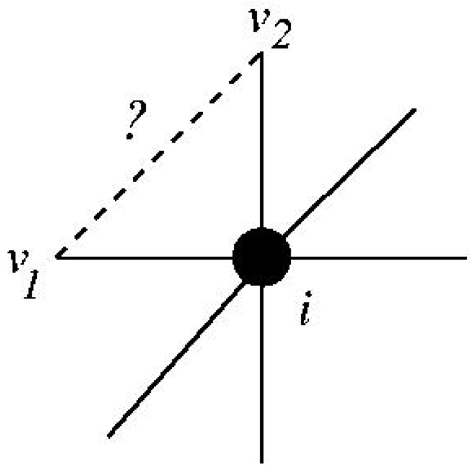
MULTIPOINT DEGREE CORRELATIONS

Measure of correlations:
$P(k',k'',\ldots k^{(n)}|k)$: conditional probability that a node of degree k is connected to nodes of degree k', k'',…

Simplest case:
$P(k'|k)$: conditional probability that a node of degree k' is connected to a node of degree k

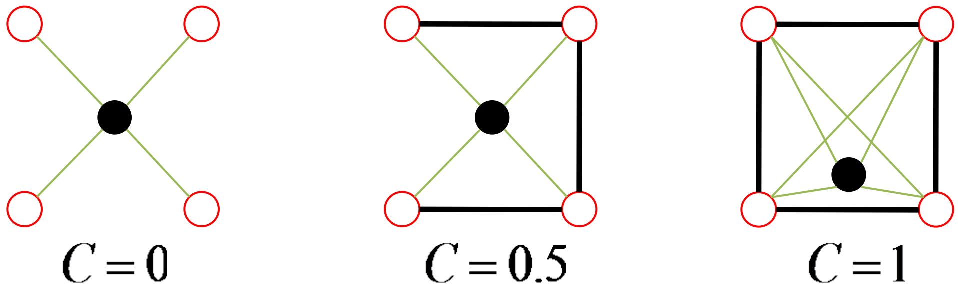
2-POINTS: *CLUSTERING COEFFICIENT*

- P(k',k''|k): cumbersome, difficult to estimate from data



Do your friends know each other ?

$$C(i) = \frac{\text{\# of links between neighbors}}{\frac{k(k-1)}{2}}$$



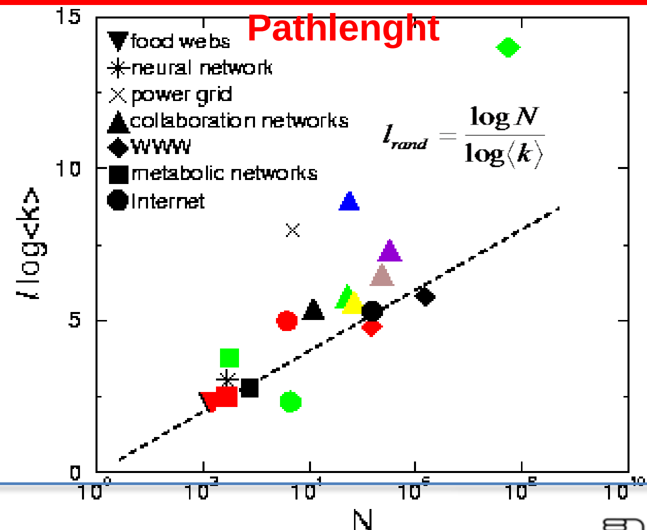
CORRELATIONS: CLUSTER SPECTRUM

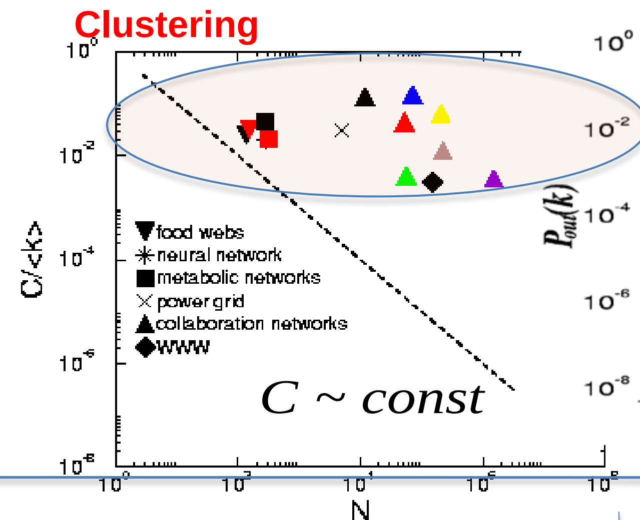
- Average clustering coefficient

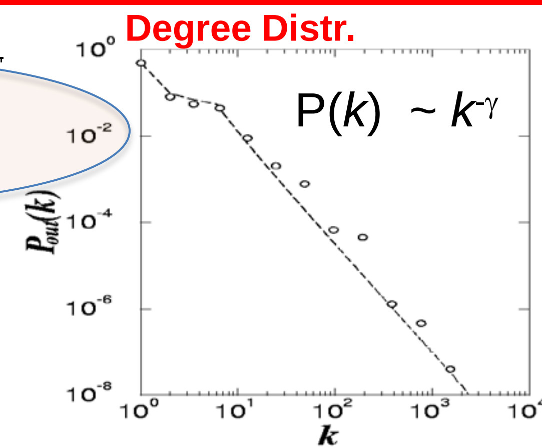
= average over nodes with very different characteristics

$$\overline{C} = \frac{1}{N}\sum_i C(i)$$

EMPIRICAL DATA FOR REAL NETWORKS







Regular network	$l \approx N^{1/D}$	$C \sim const$	$P(k)=\delta(k-k_d)$
Erdos-Renyi	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C_{rand} = p = \frac{\langle k\rangle}{N}$	$P(k) = e^{-\langle k\rangle}\frac{\langle k\rangle^k}{k!}$
Watts-Strogatz	$l_{rand} \approx \frac{\log N}{\log\langle k\rangle}$	$C \sim const$	*Exponential*
Barabasi-Albert	$l \approx \frac{\ln N}{\ln\ln N}$	$C \sim \frac{(\ln N)^2}{N}$	$P(k) \sim k^{-\gamma}$

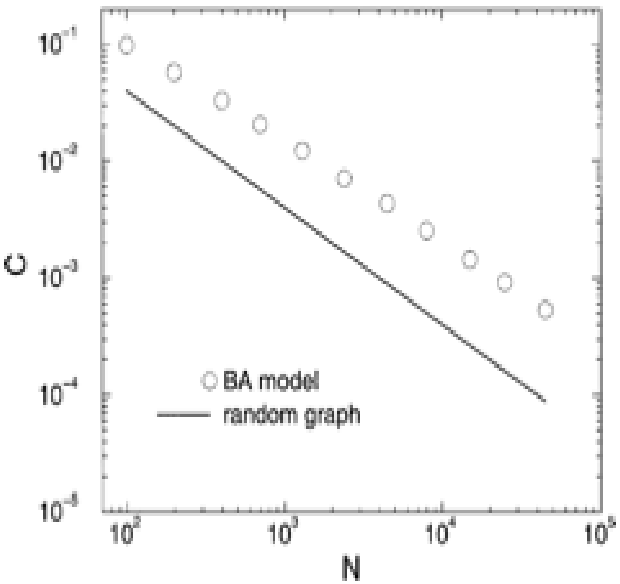
CLUSTERING COEFFICIENT OF THE BA MODEL

Reminder: for a random graph we have:

$$C_{rand} = \frac{<k>}{N} \sim N^{-1}$$

The numerical results indicate a *slightly* slower decay for BA network than for random networks.

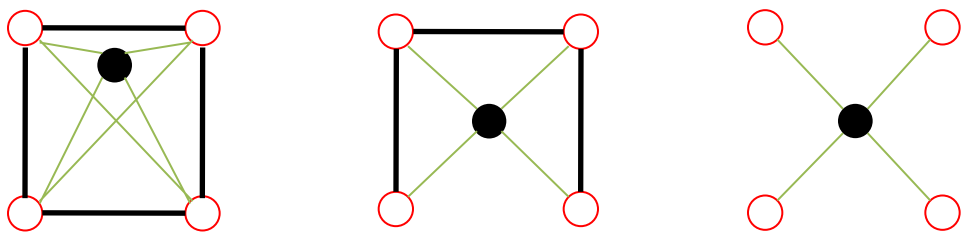
But not slow *enough*...



Clustering coefficient versus size of the Barabasi-Albert (BA) model with <k>=4, compared with clustering coefficient of random graph, $C_{rand} = \frac{<k>}{N}$

Konstantin Klemm, Victor M. Eguiluz,
Growing scale-free networks with small-world behavior,
Phys. Rev. E 65, 057102 (2002), cond-mat/0107607

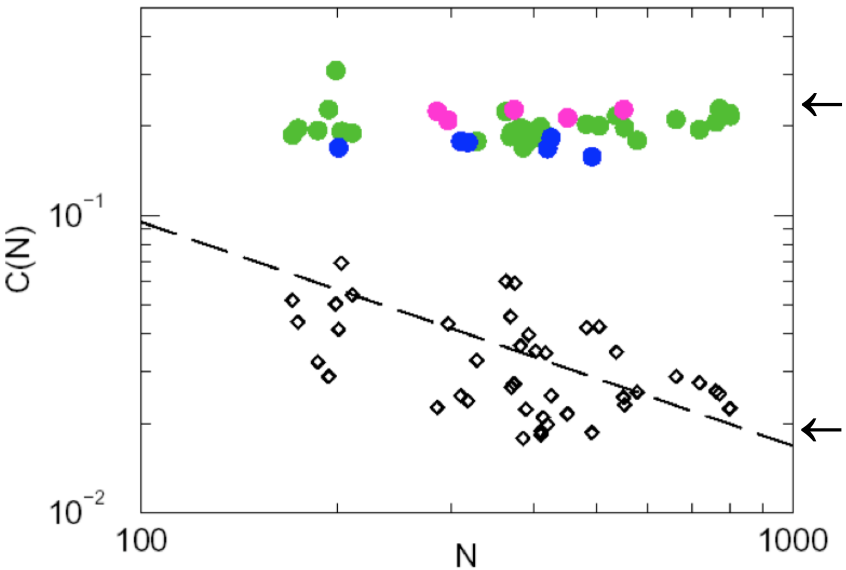
MODULARITY IN THE METABOLISM



$C=1$ $\quad$ $C=\frac{1}{2}$ $\quad$ $C=0$

Clustering Coefficient:

$$C(k) = \frac{\#\text{ links between k neighbors}}{k(k-1)/2}$$



←Metabolic network (43 organisms)

← Scale-free model