Growth and preferential attachment
The definition of the Barabási-Albert model leaves many mathematical details open:

- It does not specify the precise initial configuration of the first $m_0$ nodes.
- It does not specify whether the $m$ links assigned to a new node are added one by one, or simultaneously. This leads to potential mathematical conflicts: If the links are truly independent, they could connect to the same node $i$, leading to multi-links.

One possible definition with self-loops

$$p (i = s) = \begin{cases} \frac{k_i}{2t - 1} & \text{if } 1 \leq s \leq t - 1 \\ \frac{1}{2t - 1} & \text{if } s = t \end{cases}$$
Degree dynamics
Degree distribution

\[ k_i(t) = m \left( \frac{t}{t_i} \right)^\beta \quad \beta = \frac{1}{2} \quad \text{for } t \geq m_0+i \text{ and } 0 \text{ otherwise as system size at } t \text{ is } N=m_0+t-1 \]

A random node \( j \) at time \( t \) then is with equal probability \( 1/N=1/(m_0+t-1) \) one of the nodes 1, 2, ..., \( N \) (we assume the initial \( m_0 \) nodes create a fully connected graph), its degree will grow with the above equation, so

\[ P(k_j(t)) < k = P \left( t_j > \frac{m_0 t}{k^\beta} \right) = 1 - P \left( t_j \leq \frac{m_0 t}{k^\beta} \right) = 1 - \frac{1}{k^\beta (t + m_0 - 1)} \]

For the large times \( t \) (and so large network sizes) we can replace \( t-1 \) with \( t \) above, so

\[ \therefore P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^2 t}{m_o + t} \frac{1}{k^3} \sim k^{-3} \]

\[ \gamma = 3 \]

Degree distribution

\[ k_i(t) = m \left( \frac{t}{t_i} \right)^{\beta} \quad \beta = \frac{1}{2} \]

\[ P(k) = \frac{2m^2 t}{t - t_0} \frac{1}{k^3} \sim k^{-\gamma} \]

(i) The degree exponent is independent of \( m \).

(ii) As the power-law describes systems of rather different ages and sizes, it is expected that a correct model should provide a time-independent degree distribution. Indeed, asymptotically the degree distribution of the BA model is independent of time (and of the system size \( N \)) \( \rightarrow \) the network reaches a stationary scale-free state.

(iii) The coefficient of the power-law distribution is proportional to \( m^2 \).

Section 4

\[ \gamma = 3 \]
NUMERICAL SIMULATION OF THE BA MODEL

(a) We generated networks with \(N=100,000\) and \(m_0=m=1\) (blue), 3 (green), 5 (grey), and 7 (orange). The fact that the curves are parallel to each other indicates that \(\gamma\) is independent of \(m\) and \(m_0\). The slope of the purple line is -3, corresponding to the predicted degree exponent \(\gamma=3\). Inset: (5.11) predicts \(p_k \sim 2m^2\), hence \(p_k/2m^2\) should be independent of \(m\). Indeed, by plotting \(p_k/2m^2\) vs. \(k\), the data points shown in the main plot collapse into a single curve.

(b) The Barabási-Albert model predicts that \(p_k\) is independent of \(N\). To test this we plot \(p_k\) for \(N = 50,000\) (blue), 100,000 (green), and 200,000 (grey), with \(m_0=m=3\). The obtained \(p_k\) are practically indistinguishable, indicating that the degree distribution is stationary, i.e. independent of time and system size.
NUMERICAL SIMULATION OF THE BA MODEL

m-dependence

\[ P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \]

\[ P(k) = \frac{2m^2 t}{m_o + t} \frac{1}{k^3} \]

\[ \gamma = 3 \]

Stationarity:

\( P(k) \) independent of \( N \)

Insert:

degree dynamics

\[ k_i(t) = m \left( \frac{t}{t_i} \right)^\beta \quad \beta = \frac{1}{2} \]
The mean field theory offers the correct scaling, BUT it provides the wrong coefficient of the degree distribution.

So asymptotically it is correct \((k \rightarrow \infty)\), but not correct in details (particularly for small \(k\)).

To fix it, we need to calculate \(P(k)\) exactly, which we will do next using a rate equation based approach.
MFT - Degree Distribution: Rate Equation

\[ <N(k,t)> = tP(K,t) \]

Number of nodes with degree \( k \) at time \( t \).

Since at each timestep we add one node, we have \( N = t \) (total number of nodes = number of timesteps)

\[ \Pi(k) = \frac{k}{\sum_j k_j} = \frac{k}{2mt} \]

\( 2m \) : each node adds \( m \) links, but each link contributed to the degree of 2 nodes

Number of links added to degree \( k \) nodes after the arrival of a new node:

- Nr. of degree \( k-1 \) nodes that acquire a new link, becoming degree \( k \)
  \[ \frac{k-1}{2} P(k-1,t) \]

- Nr. of degree \( k \) nodes that acquire a new link, becoming degree \( k+1 \)
  \[ \frac{k}{2} P(k,t) \]

\[ (N + 1)P(k,t + 1) = NP(k,t) + \frac{k-1}{2} P(k-1,t) - \frac{k}{2} P(k,t) \]

# k-nodes at time \( t+1 \) # k-nodes at time \( t \) Gain of k-nodes via \( k-1 \rightarrow k \) Loss of k-nodes via \( k \rightarrow k+1 \)

MFT - Degree Distribution: Rate Equation

We do not have $k=0,1,...,m-1$ nodes in the network (each node arrives with degree $m$) → We need a separate equation for degree $m$ modes

$$ (N + 1)P(m, t + 1) = NP(m, t) + \frac{1}{2} m P(m, t) $$
We assume that there is a stationary state in the $N=t \to \infty$ limit, when $P(k,\infty) = P(k)$.

\[
(N + 1)P(k, t + 1) = NP(k, t) + \frac{k - 1}{2} P(k - 1, t) - \frac{k}{2} P(k, t)
\]

\[
(N + 1)P(m, t + 1) = NP(m, t) + 1 - \frac{m}{2} P(m, t)
\]
MFT - Degree Distribution: Rate Equation

\[ P(k) = \frac{k-1}{k+2} P(k-1) \quad \Rightarrow \quad P(k+1) = \frac{k}{k+2} P(k) \]

\[ P(m) = \frac{2}{m+2} \]

\[ P(m+1) = \frac{m}{m+3} P(m) = \frac{2m}{(m+2)(m+3)} \]

\[ P(m+2) = \frac{m+1}{m+4} P(m+1) = \frac{2m(m+1)}{(m+2)(m+3)(m+4)} \]

\[ P(m+3) = \frac{m+2}{m+5} P(m+2) = \frac{2m(m+1)}{(m+3)(m+4)(m+5)} \quad m+3 \rightarrow k \]

\[ \quad \ldots \]

\[ P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \]

\[ P(k) \sim k^{-3} \quad \text{for large } k \]

Krapivsky, Redner, Leyvraz, PRL 2000
Dorogovtsev, Mendes, Samukhin, PRL 2000
Bollobas et al, Random Struc. Alg. 2001
MFT - Degree Distribution: A Pretty Caveat

Start from eq.

\[ P(k) = \frac{k-1}{2} P(k-1) - \frac{k}{2} P(k) \]

\[ 2P(k) = (k-1)P(k-1) - kP(k) = -P(k-1) - k[P(k) - P(k-1)] \]

\[ 2P(k) = -P(k-1) - k \frac{P(k) - P(k-1)}{k - (k-1)} = -P(k-1) - k \frac{\partial P(k)}{\partial k} \]

\[ P(k) = -\frac{1}{2} \frac{\partial [kP(k)]}{\partial k} \]

Its solution is: \( P(k) \sim k^{-3} \)

Dorogovtsev and Mendes, 2003
All nodes follow the same growth law

\[ \frac{\partial k_i}{\partial t} \propto \Pi(k_i) = A \frac{k_i}{\sum_j k_j} \]

Use: \[ \sum_j k_j = 2m(t - 1) + \frac{m_0(m_0 - 1)}{2} \]

During a unit time (time step): \[ \Delta k = m \Rightarrow A = m \]

\[ \frac{\partial k_i}{\partial t} = \frac{k_i}{2t} \quad \frac{\partial k_i}{k_i} = \frac{\partial t}{2t} \quad \int_m \frac{\partial k_i}{k_i} = \int_i \frac{\partial t}{2t} \]

\[ \ln \left( \frac{k}{m} \right) = \frac{1}{2} \ln \left( \frac{t}{t_i} \right) = \ln \left( \frac{t}{t_i} \right)^{\frac{1}{2}} \]

\[ k_i(t) = m \left( \frac{t}{t_i} \right)^\beta \quad \beta = \frac{1}{2} \]

\[ \beta: \text{dynamical exponent} \]

Fitness Model
Fitness Model: Can Latecomers Make It?

**SF model:** \( k(t) \sim t^{\frac{1}{2}} \) \hspace{1cm} (first mover advantage)

**Fitness model:**

\[
\text{fitness } (\eta) \quad \Pi(k_i) \approx \frac{\eta_i k_i}{\sum_j \eta_j k_j} \quad k(\eta, t) \sim t^{\beta(\eta)}
\]

\[
\beta(\eta) = \eta / C
\]

FITNESS MODEL: Can Latecomers Make It?

**Fit-gets-rich**

**Bose-Einstein condensation**
• The degree of each node increases following a power-law with the same dynamical exponent $\beta = 1/2$ (Figure 5.6a). Hence all nodes follow the same dynamical law.

• The growth in the degree is sublinear (i.e. $\beta < 1$). This is a consequence of the growing nature of the Barabási-Albert model: Each new node has more nodes to link to than the previous node. Hence, with time the existing nodes compete for links with an increasing pool of other nodes.

• The earlier node $i$ was added, the higher is its degree $k_i(t)$. Hence, hubs are large because they arrived earlier, a phenomenon called first-mover advantage in marketing and business.

• The rate at which the node $i$ acquires new links is given by the derivative of (5.7)

$$\frac{dk_i(t)}{dt} = \frac{m}{2} \frac{1}{\sqrt{t_i t}}$$

(5.8)

indicating that in each time frame older nodes acquire more links (as they have smaller $t_i$). Furthermore the rate at which a node acquires links decreases with time as $t^{-1/2}$. Hence, fewer and fewer links go to a node.
Absence of growth and preferential attachment
\[ \Pi(k_i) : \text{uniform} \]

\[ \frac{\partial k_i}{\partial t} = A \Pi(k_i) = \frac{m}{m_0 + t - 1} \]

\[ k_i(t) = m \ln\left( \frac{m_0 + t - 1}{m + t_i - 1} \right) + m \]

\[ P(k) = \frac{e}{m} \exp\left(-\frac{k}{m}\right) \sim e^{-k} \]
\[
\frac{\partial k_i}{\partial t} = A \Pi(k_i) + \frac{1}{N} = \frac{N}{N-1} \frac{k_i}{2t} + \frac{1}{N}
\]

\[
k_i(t) = \frac{2(N-1)}{N(N-2)} t + Ct \frac{N}{2(N-1)} \sim \frac{2}{N} t
\]

\[p_k : \text{power law (initially)} \to \to \text{Gaussian} \to \text{Fully Connected}\]
Do we need both growth and preferential attachment?

YEP
**EMPIRICAL DATA FOR REAL NETWORKS**

### Pathlength
- Regular network: \( l \approx \sqrt{N/L} \)
- Erdos-Renyi: \( l_{\text{rand}} \approx \frac{\log N}{\log \langle k \rangle} \)
- Watts-Strogatz: \( l_{\text{rand}} \approx \frac{\log N}{\log \langle k \rangle} \)

### Clustering
- \( C \sim \text{const} \)
- Erdos-Renyi: \( C_{\text{rand}} = p = \frac{\langle k \rangle}{N} \)

### Degree Distribution
- \( P(k) = \delta(k - k_d) \)
- Barabasi-Albert: \( P(k) \sim k^\gamma \)
- Exponential: \( P(k) = e^{-k} \frac{<k>^k}{k!} \)

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Network Science: Evolving Network Models