3.11.5-6. Snobbish Network (textbook, p. 102)

3.11.5. Consider a network of N red and N blue nodes. The probability that there is a link between nodes of identical color is p and the probability that there is a link between nodes of different color is q. A network is snobbish if $p > q$, capturing a tendency to connect to nodes of the same color. For $q = 0$ the network has at least two components, containing nodes with the same color.

(a) Calculate the average degree of the "blue" subnetwork made of only blue nodes, and the average degree in the full network.

Answer: Each blue node has $N-1$ blue neighbors, so $N-1$ potential edges, so each blue node has on average $<k_{\text{blue}}> = (N-1)p$ edges. Likewise, each blue node has $N$ red neighbors so expected number of blue-red edges is $<k_{\text{blue-red}}>=Nq$ edges and since the same holds for red nodes, the average degrees in the blue network and the full network are

$$<k_{\text{blue}}>=Np-p$$  $$<k_{\text{full}}>=<k_{\text{blue}}>+<k_{\text{blue-red}}>=N(p+q)-p$$

(b) Determine the minimal $p$ and $q$ required to have, with high probability, just one component.

Answer: To have one component of the blue network and one component of the red network, we need to have $pN > \ln N$ since each of these networks is an ER network with $N$ nodes. To have a blue-red edge (one is sufficient), $qN/2^2 \geq 1$. Note that having $qN \geq 2$ would give us on average one blue-red edge per blue (and also red) node, so too many. Final answer:

$$p > (\ln N)/N$$  $$q \geq 2/N^2$$

(c) Show that for large $N$ even very snobbish networks ($p \gg q$) display the small-world property.

Answer: Since for $N \to \infty \ln N/N \to 0$ (because $(\ln N)' = 1/N \to 0$ and $(N)' = 1 \to 1$), then for sufficiently large $N$ $p \ln N/N$, so we will have one component of each of the one color subnetworks. For $N^2 > 1/q$ we will also
have a connection between one color subnetworks, so one giant component
by solution (b). Since ER graphs have small world property, and their
diameter \( <d> \sim \ln N \), then our full network has diameter \( <d> \sim O(2\ln N+1) \)
\( \sim O(\ln N) \). Indeed, there is at most \( \sim \ln N \) steps to get from any node to a node
that is of the same color, so the same is true for getting from any node to a
node of the same color that has an edge connecting it to the other color
subnetwork; by following this edge (one hop) we can reach any node in the
other color network also in \( \sim \ln N \) steps, so total is \( \sim 2\ln N+1 \sim \ln N \).

3.11.6. Consider the following variant of the above model, in which we have
2\(N\) nodes with equal number of blue and red nodes and fraction \(f\) of purple
nodes. Blue and red nodes do not connect to each other \((q = 0)\) while they
connect with probability \(p\) to nodes of the same color. Purple nodes connect
with the same probability \(p\) to red and blue nodes.

(a) We call the red and blue communities interactive if a typical red node
is just two steps away from a blue node and vice versa. Evaluate the fraction
of purple nodes required for the communities to be interactive.

Answer: Let \(N_p\) denotes the number of purple nodes, so \(f = N_p/(2N+N_p)\)
which means that \(2fN+fN_p=N_p\) so \(N_p(1-f)=2fN\) and therefore
\( \sum 1 \)

\[ N_p = \frac{2fN}{1-f}. \]
Overall there are be $p^2 N N_p$ edges from purple to blue nodes, hence on average each blue node will be connected to $p N_p$ purple nodes, each of which will be connected to $p N$ red nodes. To have two step connection from a blue node to red node we must have $(p N_p)(p N) \geq N$ so $p^2 N_p \geq 1$ hence (2) $N_p \geq 1/p^2$.

Plugging this into (1) we get $2fN/(1-f) \geq 1/p^2$ so $2fp^2 N \geq 1-f$ hence $f(1+2p^2 N) \geq 1$ and the result:

$$f \geq 1/(1+2p^2 N)$$

Actually, inequality (2) for size of purple community is much nicer to reason about then the final inequality bounding a fraction. It is clear that smaller $p$ larger purple community must be. With $p=1$ only one purple node is needed regardless the size of red and blue communities.

(b) Comment on the size of the purple community if the average degree of blue (or red) nodes is $<k> \geq 1$.

Answer: Let $N_p$ denotes the number of purple nodes. Then each blue or red node has on average $p(N-1)$ edges to the nodes of same color and $p N_p$ edges to purple nodes, so $p(N-1+N_p)$ edges total. Hence, the requirement that $p(N-1+N_p) \geq 1$ and therefore the final condition:

$$N_p \geq 1/p - N + 1$$

So lower the probability $p$ is, larger the size of purple community must be.