List Comprehensions and ADTs (slides, where noted, due to Graham Hutton)

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Outline

- List comprehensions
- String comprehensions
- Type declarations
- Algebraic data types (ADTs)
- Pattern matching
- Case expressions

- Countdown: putting these together

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Set Comprehensions

In mathematics, the <u>comprehension</u> notation can be used to construct new sets from old sets.

$\{x^2 \mid x \in \{1...5\}\}$

The set {1,4,9,16,25} of all numbers x^2 such that x is an element of the set {1…5}.

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Lists Comprehensions

In Haskell, a similar comprehension notation can be used to construct new <u>lists</u> from old lists.

```
[x^2 | x ← [1..5]]
```

The list [1,4,9,16,25] of all numbers x^2 such that x is an element of the list [1..5].

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Note:

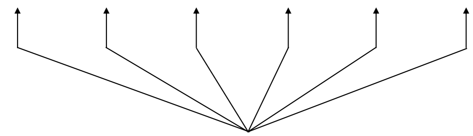
- The expression x ← [1..5] is called a generator, as it states how to generate values for x
- Comprehensions can have multiple generators, separated by commas. For example:

```
> [(x,y) | x ← [1,2,3], y ← [4,5]]

[(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)]
```

```
> [(x,y) | y ← [4,5], x ← [1,2,3]]
[(1,4),(2,4),(3,4),(1,5), ... ]
```

- Multiple generators resemble nested loops, with later generators as more deeply nested loops whose variables change value more frequently


```
> [(x,y) | y ← [4,5], x ← [1,2,3]]

[(1,4),(2,4),(3,4),(1,5),(2,5),(3,5)]
```



x ← [1,2,3] is the last generator, so the value of the x component of each pair changes most frequently.

Dependant Generators

Later generators can depend on the variables that are introduced by earlier generators.

```
> [(x,y) | x ← [1..3], y ← [x..3]]
[(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)]
```


Using a dependant generator can you define the library function that concatenates a list of lists:

For example:

```
> concat [[1,2,3],[4,5],[6]]

[1,2,3,4,5,6]
```

```
concat :: [[a]] → [a]
concat xss = [x | xs ← xss, x ← xs]
```

Guards

List comprehensions can use guards to restrict the values produced by earlier generators.

```
[x | x ← [1..10], even x]
```

The list [2,4,6,8,10] of all numbers x such that x is an element of the list [1..10] and x is even.

Can you define a function that maps a positive integer to its list of factors, e.g.,

```
> factors 15
[1,3,5,15]
```

```
factors :: Int → [Int]
factors n = [x | x ← [1..n], n `mod` x == 0]
```

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Using factors, we can define a function that decides if a number is prime? E.g.,

prime n = length (factors n) == 2

```
> prime 15
False

> prime 7
True
```

```
prime :: Int → Bool
prime n = factors n == [1,n]
```

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And using prime as guard, we can now define a function that returns the list of all primes up to a given limit. E.g.,:

```
> primes 40

[2,3,5,7,11,13,17,19,23,29,31,37]
```

```
primes :: Int → [Int]
primes n = [x | x <- [2..n], prime x]
```

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The Zip Function

A useful library function is zip, which maps two lists to a list of pairs of their corresponding elements.

```
zip :: [a] → [b] → [(a,b)]
```

For example:

```
> zip ['a','b','c'] [1,2,3,4]

[('a',1),('b',2),('c',3)]
```

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Using zip we can define a function that returns the list of all pairs of adjacent elements from a list:

For example:

[1,2,3,4]
[2,3,4]

```
> pairs [1,2,3,4]

[(1,2),(2,3),(3,4)]
```

```
pairs :: [a] → [(a,a)]
pairs xs = zip xs (tail xs)
```


Using pairs, we can define a function that decides if the elements in a list are sorted. For example:

```
> sorted [1,2,3,4]
True

> sorted [1,3,2,4]
False
```

```
sorted :: Ord a ⇒ [a] → Bool
sorted xs = and [x ≤ y | (x,y) ← pairs xs]
```

[True, False, True]

Using zip we can define a function that returns the list of all positions of a value in a list. For example:

```
> positions 0 [1,0,0,1,0,1,1,0]
[1,2,4,7]
```

```
positions :: Eq a ⇒ a → [a] → [Int]
positions x xs =
   [i | (x',i) ← zip xs [0..], x == x']
```


String Comprehensions

A string is a sequence of characters enclosed in double quotes. Internally, however, strings are represented as lists of characters

```
"abc" :: String
```

Means [' a' , ' b' , ' c'] :: [Char].

Because strings are just special kinds of lists, any polymorphic function that operates on lists can also be applied to strings. For example:

```
> length "abcde"
5

> take 3 "abcde"
"abc"

> zip "abc" [1,2,3,4]
[('a',1),('b',2),('c',3)]
```


Similarly, list comprehensions can also be used to define functions on strings, such counting how many times a character occurs in a string.

For example:

```
> count 's' "Mississippi"
4
```

```
count :: Char → String → Int
count x xs = length [x' | x' ← xs, x == x']
```


Exercises

(1) A triple (x,y,z) of positive integers is called pythagorean if $x^2 + y^2 = z^2$. Using a list comprehension, define a function

```
pyths :: Int → [(Int,Int,Int)]
```

that maps an integer n to all such triples with components in [1..n]. For example:

```
> pyths 5
[(3,4,5),(4,3,5)]
```


(2) A positive integer is perfect if it equals the sum of all of its factors, excluding the number itself. Using a list comprehension, define a function

```
perfects :: Int → [Int]
```

that returns the list of all perfect numbers up to a given limit. For example:

```
> perfects 500

[6,28,496]
```

[1, 2, 3] [4, 5, 6]
= 32

(3) The inner product of two lists of integers xs and ys of length n is given by the sum of the products of the corresponding integers:

$$\sum_{i=0}^{n-1} (xs_i * ys_i)$$

zipWith

Hoogle

Using a list comprehension, define a function that returns the inner product of two lists.

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- String comprehensions
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- Algebraic datatypes (ADTs)
- Pattern matching
- Case expressions
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Type Declarations

In Haskell, a new name for an existing type can be defined using a type declaration.

```
type String = [Char]
```

String is a synonym, also called *type alias*, for the type [Char].

Type declarations can be used to make other types easier to read. For example, given

```
type Pos = (Int,Int)
```

we can define:

```
origin :: Pos
origin = (0,0)

left :: Pos → Pos
left (x,y) = (x-1,y)
```

Like function definitions, type declarations can also have parameters. For example, given

```
type Pair a = (a,a)
```

we can define:

```
mult :: Pair Int → Int
mult (m,n) = m * n

copy :: a → Pair a
copy x = (x,x)
```

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Type declarations can be nested:

```
type Pos = (Int,Int)

type Trans = Pos → Pos
```

✔

However, they cannot be recursive:

```
type Tree = (Int,[Tree])
```

✘

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Algebraic Datatypes (ADTs)

A completely new type, an algebraic data type, can be defined by specifying its values using a data declaration.

```
data Bool = False | True
```

Bool is a new type, with two new values False and True.

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Note:

- The two values False and True are called the constructors for the datatype Bool
- Datatype and constructor names must always begin with an upper-case letter
- The Bool datatype is an enumeration type – this is Haskell's way of defining them. Other examples:

```
data Day = Mon | Tue | Wed | Thu | Fri | Sat | Sun
data Month = Jan | Feb | … | Nov | Dec
```

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Values of those new types and datatypes can be used in the same ways as those of built in types. For example, given

```
data Answer = Yes | No | Unknown
```

we can define:

```
answers :: [Answer]
answers = [Yes,No,Unknown]

flip :: Answer → Answer
flip Yes     = No
flip No      = Yes
flip Unknown = Unknown
```

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The constructors in a datatype declaration can also have parameters. For example, given

```
data Shape = Circle Float
           | Rect Float Float
```

we can define:

```
square :: Float → Shape
square n = Rect n n

area :: Shape → Float
area (Circle r) = pi * r^2
area (Rect x y) = x * y
```

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Note:

- Shape has values of the form Circle r where r is a float, and Rect x y where x and y are floats.
- Circle and Rect can be viewed as <u>functions</u> that construct values of type Shape:

```
Circle :: Float → Shape

Rect :: Float → Float → Shape
```

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Algebraic data types are tagged unions (aka sums) of products (aka records)

```
data Shape = Circle Float
           | Rect Float Float
```

union

Haskell keyword

the new type

new constructors (a.k.a. tags, disjuncts, summands) ~~Line~~ is a ~~binary~~ constructor, ~~Triangle~~ is a ~~ternary~~ …

Circle unary Rect binary

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Not surprisingly, data declarations themselves can also have parameters. For example, given

```
data Maybe a = Nothing | Just a
```

we can define: safediv

10 `div` 5 = 2
10 `div` 0 = ...

safediv :: Int -> Int -> Maybe Int
safediv _ 0 = Nothing
safediv n d = Just (n `div` d)

Recursive Types

In Haskell, algebraic datatypes can be declared in terms of themselves. That is, they can be recursive.

```
data Nat = Zero | Succ Nat
```

Nat is a new type, with constructors Zero :: Nat and Succ :: Nat → Nat.

0 : Zero
1 : Succ Zero
2 : Succ (Succ Zero)
...

Note:

- A value of type Nat is either Zero, or of the form Succ n where n :: Nat. That is, Nat contains the following infinite sequence of values:

```
Zero

Succ Zero

Succ (Succ Zero)

⋮
```


- We can think of values of type Nat as natural numbers, where Zero represents 0, and Succ represents the successor function 1+.

- For example, the value

```
Succ (Succ (Succ Zero))
```

represents the natural number

```
1 + (1 + (1 + 0)) = 3
```

Using recursion, it is easy to define functions that convert between values of type Nat and Int:

```
nat2int :: Nat → Int
nat2int Zero     = 0
nat2int (Succ n) = 1 + nat2int n

int2nat :: Int → Nat
int2nat 0 = Zero
int2nat n = Succ (int2nat (n-1))
```


data Nat = Zero | Succ Nat

Two naturals can be added by converting them to integers, adding, and then converting back:

```
add :: Nat → Nat → Nat
add m n = int2nat (nat2int m + nat2int n)
```

However, using recursion the function add can be defined without the need for conversions:

```
add Zero     n = n
add (Succ m) n = Succ (add m n)
```


data Lexp = Var Name
| App Lexp Lexp
| Lambda Name Lexp

Using recursion, a suitable new algebraic datatype to represent arithmetic expressions can be declared by:

```
data Expr = Val Int
          | Add Expr Expr
          | Mul Expr Expr
```

For example, a certain expression is represented as follows:

1 2*3

```
Add (Val 1) (Mul (Val 2) (Val 3))
```

1 + 2*3

Using recursion, it is now easy to define functions that process expressions. For example:

```
size :: Expr → Int
size (Val n)   = 1
size (Add x y) = size x + size y
size (Mul x y) = size x + size y

eval :: Expr → Int
eval (Val n)   = n
eval (Add x y) = eval x + eval y
eval (Mul x y) = eval x * eval y
```

Add / Val 1 / Mul / Val 2 / Val 3

A data constructor constructs a new data object. For example:

```
Val 1
Val 3
Mul (Val 1) (Val 3)
Add (Val 1) (Mul (Val 2) (Val 3))
```

A type constructor constructs a new type object. For example:

```
safediv :: Int → Int → Maybe Int
safediv _ 0 = Nothing
safediv m n = Just (m `div` n)
```


Exercises

(1) Using recursion and the function add, define a function that multiplies two natural numbers.

(2) Define a type Tree a of binary trees built from Leaf values of type a using a Node constructor that takes two binary trees as parameters.

data Tree a = Leaf a | Node (Tree a) (Tree a)

data Tree a = Leaf | Node (Tree a) a (Tree a)

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Pattern Matching with Case Expressions

Examine values of an algebraic data type. For example:

```
area :: Shape -> Float
area s = case s of
         Circle r -> pi * r^2
         Rect x y -> x * y
```

Two important points on case expressions:

- Test: does the given value match this pattern?
- Binding: if value matches, bind corresponding values of **s** and pattern

Notes:

An underscore _ is the wildcard pattern. It matches anything.

As-pattern

A pattern x@pat can be used to match a pattern while retaining the variable referring the object being matched:

```
foo::Expr -> String
foo e@(Add _ _) = e ++ "is Add expression"
```

Patterns can be nested:

```
bar::Expr -> String
bar e@(Add _ r@(Add _ _)) = e ++ "is …"
```


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Notes:

Patterns are matched in turn until

```
foo::Expr -> String
foo e@(Add _ _) = e ++ "is Add expression"
```

Patterns can be nested:

```
bar::Expr -> String
bar e@(Add _ r@(Add _ _)) = e ++ "is …"
```


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Exercises

data Nat = Zero | Succ Nat

(1) Write the ~~safediv~~ int2Nat function using a case expression.

```
int2nat :: Int -> Nat
int2nat 0 = Zero
int2nat n = Succ (int2nat (n-1))

int2nat :: Int -> Maybe Nat
int2nat 0 = Just Zero
int2nat n = if n < 0
            then Nothing
            else Just (Succ (int2nat (n-1)))
```

Almost!

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What is Countdown? Example

Using the numbers

1 3 7 10 25 50

and the arithmetic operators

\+ - * ÷

construct an expression whose value is 765

Rules

- All the numbers, including intermediate results, must be positive naturals (1,2,3,…).
- Each of the source numbers can be used at most once when constructing the expression.
- We abstract from other rules that are adopted on television for pragmatic reasons.

For our example, one possible solution is

```
(25-10) * (50+1)   =   765
```

Notes:

- There are 780 solutions for this example.
- Changing the target number to 831 gives an example that has no solutions.

Evaluating Expressions

Operators:

```
data Op = Add | Sub | Mul | Div
```

Apply an operator:

```
apply :: Op → Int → Int → Int
apply Add x y = x + y
apply Sub x y = x - y
apply Mul x y = x * y
apply Div x y = x `div` y
```

Decide if the result of applying an operator to two positive natural numbers is another positive natural number:

```
valid :: Op → Int → Int → Bool
valid Add _ _ = True
valid Sub x y = x > y
valid Mul _ _ = True
valid Div x y = x `mod` y == 0
```

Expressions:

```
data Expr = Val Int | App Op Expr Expr
```

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Return the overall value of an expression, provided that it is a positive natural number:

```
eval :: Expr → [Int]
eval (Val n)     = [n | n > 0]
eval (App o l r) = [apply o x y | x ← eval l
                                , y ← eval r
                                , valid o x y]
```

Either succeeds and returns a singleton list or fails and returns the empty list.

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Formalising The Problem

Return a list of all possible ways of choosing zero or more elements from a list:

```
choices :: [a] → [[a]]
```

For example:

```
> choices [1,2]

[[],[1],[2],[1,2],[2,1]]
```

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Return a list of all the values in an expression:

```
values :: Expr → [Int]
values (Val n)     = [n]
values (App _ l r) = values l ++ values r
```

Decide if an expression is a solution for a given list of source numbers and a target number:

```
solution :: Expr → [Int] → Int → Bool
solution e ns n = elem (values e) (choices ns)
                  && eval e == [n]
```

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Brute Force Solution

Return a list of all possible ways of splitting a list into two non-empty parts:

```
split :: [a] → [([a],[a])]
```

For example:

```
> split [1,2,3,4]

[([1],[2,3,4]),([1,2],[3,4]),([1,2,3],[4])]
```

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Return a list of all possible expressions whose values are precisely a given list of numbers:

```
exprs :: [Int] → [Expr]
exprs []  = []
exprs [n] = [Val n]
exprs ns  = [e | (ls,rs) ← split ns
               , l       ← exprs ls
               , r       ← exprs rs
               , e       ← combine l r]
```

The key function in this lecture.

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Combine two expressions using each operator:

```
combine :: Expr → Expr → [Expr]
combine l r =
     [App o l r | o ← [Add,Sub,Mul,Div]]
```

Return a list of all possible expressions that solve an instance of the countdown problem:

```
solutions :: [Int] → Int → [Expr]
solutions ns n = [e | ns' ← choices ns
                    , e   ← exprs ns'
                    , eval e == [n]]
```

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Can We Do Better?

- Many of the expressions that are considered will typically be invalid - fail to evaluate.
- For our example, only around 5 million of the 33 million possible expressions are valid.
- Combining generation with evaluation would allow earlier rejection of invalid expressions.

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Fusing Two Functions

Valid expressions and their values:

```
type Result = (Expr,Int)
```

We seek to define a function that fuses together the generation and evaluation of expressions:

```
results :: [Int] → [Result]
results ns = [(e,n) | e ← exprs ns
                    , n ← eval e]
```

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This behaviour is achieved by defining

```
results []  = []
results [n] = [(Val n,n) | n > 0]
results ns  =
   [res | (ls,rs) ← split ns
        , lx      ← results ls
        , ry      ← results rs
        , res     ← combine' lx ry]
```

where

```
combine' :: Result → Result → [Result]
```

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Combining results:

```
combine' (l,x) (r,y) =
   [(App o l r, apply o x y)
      | o ← [Add,Sub,Mul,Div]
      , valid o x y]
```

New function that solves countdown problems:

```
solutions' :: [Int] → Int → [Expr]
solutions' ns n =
   [e | ns'   ← choices ns
      , (e,m) ← results ns'
      , m == n]
```

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Can We Do Better?

- Many expressions will be essentially the same using simple arithmetic properties, such as:

```
x * y  =  y * x

x * 1  =  x
```

- Exploiting such properties would considerably reduce the search and solution spaces.

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Exploiting Properties

Strengthening the valid predicate to take account of commutativity and identity properties:

```
valid :: Op → Int → Int → Bool
valid Add x y  = x ≤ y
valid Sub x y  = x > y
valid Mul x y  = x ≤ y && x ≠ 1 && y ≠ 1
valid Div x y  = x `mod` y == 0 && y ≠ 1
```