Types and Type Based Analysis: Lambda Calculus, Intro to Haskell

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Announcements

- Welcome back!
- HW5 is out
- Grades. I am still grading HW4
- Moving on with Types and Type-based Analysis

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Outline

- Pure lambda calculus, a review
  - Syntax and semantics (last time)
  - Free and bound variables (last time)
  - Substitution (last time)
  - Rules (last time)
  - Normal forms
  - Reduction strategies
- Interpreters for the Lambda calculus
- Coding them in Haskell

Syntax of Pure Lambda Calculus

- λ-calculus formulae (e.g., λ**x. x y**) are called expressions or terms

- **E ::= x | (λx. E_1) | (E_1 E_2)**
  - A λ-expression is one of
    - Variable: **x**
    - Abstraction (i.e., function definition): λ**x.** E_1
    - Application: E_1 E_2

Syntactic Conventions

- Parentheses may be dropped from "stand-alone" terms **(E_1 E_2)** and **(λx. E)**
  - E.g., **(f x)** may be written as **f x**

- Function application groups from left-to-right (i.e., <u>it is left-associative</u>)
  - E.g., **x y z** abbreviates **((x y) z)**
  - E.g., **E_1 E_2 E_3 E_4** abbreviates **(((E_1 E_2) E_3) E_4)**
  - Parentheses in **x (y z)** are necessary! Why?

$(\lambda x. x x)(\lambda y. y)(\lambda z. z z) \equiv ((\lambda x. x x)(\lambda y. y))(\lambda z. z z) \to^* \lambda z. z z.$

NOT $(\lambda x. x x)((\lambda y. y)(\lambda z. z z))$

Syntactic Conventions

- Application <u>has higher precedence</u> than abstraction
  - Another way to say this is that the scope of the dot extends as far to the right as possible
  - E.g., **λx. x z** = **λx. (x z)** = **(λx. (x z))** = **(λx. (x z))** ≠ **((λx. x) z)**

- WARNING: This is the most common syntactic convention (e.g., Pierce 2002). However, some books give abstraction higher precedence; you might have seen that different convention

Rules (Axioms) of Lambda Calculus

- α rule (α-conversion): renaming of parameter (choice of parameter name does not matter)
  - **λx. E** $\rightarrow_\alpha$ **λz. (E[z/x])** provided **z** is not free in **E**
  - e.g., **λx. x x** is the same as **λz. z z**
- β rule (β-reduction): function application (substitutes argument for parameter)
  - **(λx.E) M** $\rightarrow_\beta$ **E[M/x]**

  Note: **E[M/x]** as defined in class last time
  - e.g., **(λx. x) z** $\rightarrow_\beta$ **z**

Rules of Lambda Calculus: Exercises

Eta-reduction:

$\lambda x.\ M\ x \rightarrow_\eta M$ provided x does not occur as free variable in M.

Anticipate $M \equiv \lambda z. N$, i.e. M is a function value.

Then $\lambda x. M\ x \equiv \lambda x.\ \underline{(\lambda z.N)\ x} \rightarrow \lambda x.\ N[x/z] \equiv M$

($\lambda x. N[x/z]$ is α-rename of z in N)

- Reduce

(λx. x) y → ~~?~~ y

(λx. x) (λy. y) → ? (λy.y)

(λx.λy.λz. x z (y z)) (λu. u) (λv. v) → ?

Reductions

- An expression **(λx.E) M** is called a redex (for reducible expression)
- An expression is in normal form if it cannot be β-reduced
- The normal form is the meaning of the term, the "answer"

Definitions of Normal Form

- Normal form (NF): a term without redexes
- Head normal form (HNF)
  - **x** is in HNF
  - **(λx. E)** is in HNF if **E** is in HNF
  - **(x E_1 E_2 … E_n)** is in HNF

$\underbrace{x}_{E_1}\ \underbrace{((\lambda y.y)(\lambda z.z))}_{E_2}$
- Weak head normal form (WHNF)
  - **x** is in WHNF
  - **(λx. E)** is in WHNF
  - **(x E_1 E_2 … E_n)** is in WHNF

Questions

- λ**z. z z** is in NF, HNF, or WHNF? NF. NF ⇒ HNF ⇒ WHNF
- **(**λ**z. z z) (**λ**x. x)** is in? Neither
- λ**x.**λ**y.**λ**z. x z (y (**λ**u. u))** is in? NF
- $\underbrace{(\lambda x.\lambda y.\ x)}_{E_1}\ \underbrace{z}_{E_2}\ \underbrace{((\lambda x.\ z\ x)\ (\lambda x.\ z\ x))}_{E_3}$ is in? Neither
- **z ((**λ**x. z x) (**λ**x. z x))** is in? HNF and WHNF
- **(**λ**z.(**λ**x.**λ**y. x) z ((**λ**x. z x) (**λ**x. z x)))** is in?

WHNF

Simple Reduction Exercise

- **C =** λ**x.**λ**y.**λ**f. f x y**
- **H =** λ**f. f (**λ**x.**λ**y. x)** **T =** λ**f. f (**λ**x.**λ**y. y)**
- What is **H (C a b)**?

→ (λf. f (λx.λy. x)) (C a b)

→ (C a b) (λx.λy. x)

→ ((λx.λy.λf. f x y) a b) (λx.λy. x)

→ (λf. f a b) (λx.λy. x)

→ (λx.λy. x) a b

→ a

Exercise

An expression with no free variables is called combinator. S, I, C, H, T are combinators.

- **S = λx.λy.λz. x z (y z)**
- **I = λx. x**
- What is **S I I I**?

Reducible expression is underlined at each step.

(λx.λy.λz. x z (y z)) I I I

→ **(λy.λz. I z (y z)) I I**

→ **(λz. I z (I z)) I**

→ **I I (I I) = (λx. x) I (I I)**

→ **I (I I) = (λx. x) (I I)**

→ **I I = (λx. x) I → I**

Aside: Trace Semantics

- Models a trace of program execution
- In the imperative world
  - Basic operation: assignment statement
  - Execution (transition system) is a sequence of state transitions

$(\ell_1, \sigma_1) \to (\ell_2, \sigma_2) \to \ldots (\ell_{EXIT}, \sigma_{EXIT})$

ANSWER

- Assignment: ℓ_j : **x** = E; ℓ_i : …

$(\ell_j, \sigma) \to (\ell_i, \sigma[x \leftarrow [\![E]\!](\sigma)])$

- Assignment: ℓ_j : **x** = E_1 *Op* E_2 ; ℓ_i : …

$(\ell_j, \sigma) \to (\ell_i, \sigma[x \leftarrow [\![E_1]\!](\sigma) \text{ Op } [\![E_2]\!](\sigma)])$

Aside: Trace Semantics

- In the functional world $E \to E_1 \to E_2 \to \ldots$ $\underbrace{NF}_{\text{ANSWER}}$
  - Basic operation is function application
  - Execution (transition system) is a sequence of β-reductions

$\underline{(\lambda x.\lambda y.\lambda z.\ x\ z\ (y\ z))}$ I I I

→ **$\underline{(\lambda y.\lambda z.\ I\ z\ (y\ z))}$ I I**

→ **$\underline{(\lambda z.\ I\ z\ (I\ z))\ I}$**

…

$\underline{\lambda x.x}$

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- Pure lambda calculus, a review
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  - Normal forms
  - Reduction strategies
- Lambda calculus interpreters
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Reduction Strategy

S I I

- Let us look at $(\lambda x.\lambda y.\lambda z.\ x\ z\ (y\ z))\ (\lambda u.\ u)\ (\lambda v.\ v)$

NOT A REDEX!

- Actually, there are (at least) two "reduction paths":

Path 1: $(\lambda x.\lambda y.\lambda z.\ x\ z\ (y\ z))\ (\lambda u.\ u)\ (\lambda v.\ v) \rightarrow_\beta$
$(\lambda y.\lambda z.\ (\lambda u.\ u)\ z\ (y\ z))\ (\lambda v.\ v) \rightarrow_\beta$
$(\lambda z.\ (\lambda u.\ u)\ z\ ((\lambda v.\ v)\ z)) \rightarrow_\beta (\lambda z.\ z\ ((\lambda v.\ v)\ z)) \rightarrow_\beta$
$\lambda z.\ z\ z$

Path 2: $(\lambda x.\lambda y.\lambda z.\ x\ z\ (y\ z))\ (\lambda u.\ u)\ (\lambda v.\ v) \rightarrow_\beta$
$(\lambda y.\lambda z.\ (\lambda u.\ u)\ z\ (y\ z))\ (\lambda v.\ v) \rightarrow_\beta$
$(\lambda y.\lambda z.\ z\ (y\ z))\ (\lambda v.\ v) \rightarrow_\beta (\lambda z.\ z\ ((\lambda v.\ v)\ z)) \rightarrow_\beta$
$\lambda z.\ z\ z$

Reduction Strategy

AP:

NORM:

- A reduction strategy (also called evaluation order) is a strategy for choosing redexes
  - How do we arrive at the normal form (answer)?
- Applicative order reduction chooses the leftmost-innermost redex in an expression
  - Also referred to as call-by-value reduction
- Normal order reduction chooses the leftmost-outermost redex in an expression
  - Also referred to as call-by-name reduction

Reduction Strategy: Examples

- Evaluate **(λx. x x) ((λy. y) (λz. z))**
- Using applicative order reduction:

$(\lambda x.\, x\, x)(\lambda z.z) \rightarrow$

$(\lambda z.z)(\lambda z.z) \rightarrow$

$\lambda z.z$

- Using normal order reduction

$((\lambda y.y)(\lambda z.z))((\lambda y.y)(\lambda z.z)) \rightarrow$

$(\lambda z.z)((\lambda y.y)(\lambda z.z))$

$(\lambda y.y)(\lambda z.z) \rightarrow \lambda z.z$

$E' \leftarrow AP(E)$

$x : x$

$\lambda x.E : \lambda x.\, AP(E)$

$E_1\, E_2 : E_1' \leftarrow AP(E_1)$

$E_2' \leftarrow AP(E_2)$

if E_1' is $\lambda x.E_3$

$AP(E_3[E_2'/x])$

else

$E_1'\, E_2'$

Reduction Strategy

- In our examples, both strategies produced the same result. This is not always the case
  - First, look at expression **(λx. x x) (λx. x x)**. What happens when we apply β-reduction to this expression?
  - Then look at **(λz.y) ((λx. x x) (λx. x x))**
    - Applicative order reduction – what happens?
    - Normal order reduction – what happens?

Church-Rosser Theorem

- Normal form implies that there are no more reductions possible
- Church-Rosser Theorem, informally
  - If normal form exists, then it is unique (i.e., result of computation does not depend on the order that reductions are applied; i.e., no expression can have two distinct normal forms)
  - If normal form exists, then normal order will find it

Reduction Strategy

- Intuitively:
- Applicative order (call-by-value) is an eager evaluation strategy. Also known as strict
- Normal order (call-by-name) is a lazy evaluation strategy
- What order of evaluation do most PLs use?

Exercises

- Evaluate **(λx.λy. x y) ((λz. z) w)**
- Using applicative order reduction
- Using normal order reduction

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Interpreters

- An interpreter for the lambda calculus is a program that reduces lambda expressions to "answers"
- We must specify NF, WHNF, HNF?
  - The definition of "answer". Which normal form?
  - The reduction strategy. How do we choose redexes in an expression? AP, NORM

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An Interpreter

Haskell syntax:
let …. in
case f of
->

- Definition by cases on $E ::= x \mid \lambda x.\ E_1 \mid E_1\ E_2$

interpret(**x**) = **x**

interpret(λ**x.E**$_1$) = λ**x.E**$_1$ WHNF

interpret(**E**$_1$ **E**$_2$) = let **f** = interpret(**E**$_1$)

in case **f** of

λ**x.E**$_3$ -> interpret(**E**$_3$[**E**$_2$/**x**]) NORM

\- -> **f E**$_2$

- What normal form: Weak head normal form
- What strategy: Normal order

Another Interpreter

- Definition by cases on $E ::= x \mid \lambda x.\ E_1 \mid E_1\ E_2$

interpret(**x**) = **x**

interpret(λ**x.E**$_1$) = λ**x.E**$_1$ WHNF

interpret(**E**$_1$ **E**$_2$) = let **f** = interpret(**E**$_1$)

a = interpret(**E**$_2$)

in case **f** of

λ**x.E**$_3$ → interpret(**E**$_3$[**a**/**x**]) AP

\- → **f a**

- What normal form: Weak head normal form
- What strategy: Applicative order

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Coding them in Haskell

- In HW5 you will code an interpreter in Haskell
- Haskell
  - A functional programming language
- Key ideas
  - Lazy evaluation
  - Static typing and polymorphic type inference
  - Algebraic data types and pattern matching
  - Monads … and more

Lazy Evaluation

- Unlike Scheme (and most programming languages) Haskell does lazy evaluation, i.e., normal order reduction
  - It won't evaluate an argument expr. until it is needed

\> **f x = []** // **f** takes **x** and returns the empty list

\> **f (repeat 1)** // returns? map (\x -> (show x) ++ "_") [1..]

["1_", "2_", "3_", ...]

\> **[]** [2..]

\> **head (tail [1..])** // returns?

\> **2** // **[1..]** is infinite list of integers

- Lazy evaluation allows us to work with infinite structures!

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Static Typing and Type Inference

- Unlike Scheme, which is dynamically typed, Haskell is statically typed!
- Unlike Java/C++ we don't always have to write type annotations. Haskell infers types!
  - A lot more on type inference later!

\> **f x = head x** // **f** returns the head of list **x**

\> **f True** // returns? :t f

\> f :: [a] -> a

• Couldn't match expected type '[a]' with actual type 'Bool'

• In the first argument of 'f', namely 'True'

In the expression: f True …

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Algebraic Data Types

- Algebraic data types are tagged unions (aka sums) of products (aka records)

```
data Shape = Line Point Point
           | Triangle Point Point Point
           | Quad Point Point Point Point
```

union

Haskell keyword

the new type

new constructors (a.k.a. tags, disjuncts, summands)
Line is a binary constructor, Triangle is a ternary …

Algebraic Data Types in HW5

- Constructors create new values
- Defining a lambda expression

```
type Name = String
data Expr = Var Name
          | Lambda Name Expr
          | App Expr Expr
```

> e1 = Var "x" // Lambda term **x**

> e2 = Lambda "x" e1 // Lambda term **λx.x**

Examples of Algebraic Data Types

Polymorphic types.
a is a type parameter!

data Bool = True | False

data Day = Mon | Tue | Wed | Thu | Fri | Sat | Sun

data List **a** = Nil | Cons **a** (List **a**)

data Tree **a** = Leaf **a** | Node (Tree **a**) (Tree **a**)

[data Maybe **a** = Nothing | Just **a**] Optional [Nut]

Maybe type denotes that result of computation can be **a** or Nothing. Maybe is a monad.

Data Constructors vs Type Constructors

- Data constructor constructs a "program object"
  - E.g., Var, Lambda, and App are data constructs
- Type constructor constructs a "type object"
  - E.g., Maybe is a unary type constructor

Pattern Matching

Type signature of anchorPnt: takes a Shape and returns a Pnt.

- Examine values of an algebraic data type

```
anchorPnt :: Shape → Pnt
anchorPnt s = case s of
                Line      p1 p2 → p1
                Triangle p3 p4 p5 → p3
                Quad      p6 p7 p8 p9 → p6
```

- Two points
  - Test: does the given value match this pattern?
  - Binding: if value matches, bind corresponding values of **s** and pattern

Pattern Matching in HW5

```
isFree::Name → Expr → Bool
isFree v e =
    case e of
        Var n → if (n == v) then True else False
        Lambda …
```

Type signature of **isFree**. In Haskell, all functions are curried, i.e., they take just one argument. **isFree** takes a variable name, and returns a function that takes an expression and returns a boolean.

Of course, we can interpret **isFree** as a function that takes a variable name **name** and an expression **E**, and returns true if variable **name** is free in **E**.

Haskell Resources

- http://www.seas.upenn.edu/~cis194/spring13/
- https://www.haskell.org/