Mathematical Recreations at the Casino

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Introduction

This presentation was inspired by Connolly's article [1] on the Ultimate Strategy for casino gambling. In this note we explore a topic of great interest to most gamblers. What is the optimal betting strategy for the following simple problem?

1. A Simple Win/Loss Betting Game

A gambler starts with some amount of money c_0 and he will play at the casino until he goes broke or until he attains his goal of T dollars. On each play he makes a bet b, wins b with probability p and loses his bet with probability $1-p$. The gambler can never bet more than his current wealth c, and it stands to reason that he will never bet more than $T-c$. Furthermore, he must have a fixed strategy, that is, any time he finds himself with wealth c, he will make the same bet, which we denote $B(c)$. We consider a strategy to be optimal if it maximizes the probability that the gambler leaves the casino having achieved his target T.

Let c_t denote the gambler's wealth after he has played t rounds. Using a betting strategy B, the gambler's worth is a random process that obeys the following update rule.

$$c_{t+1} = \begin{cases} c_t + B(c_t) & \text{with probability } \; p \\ c_t - B(c_t) & \text{with probability } \; 1-p \end{cases} .$$

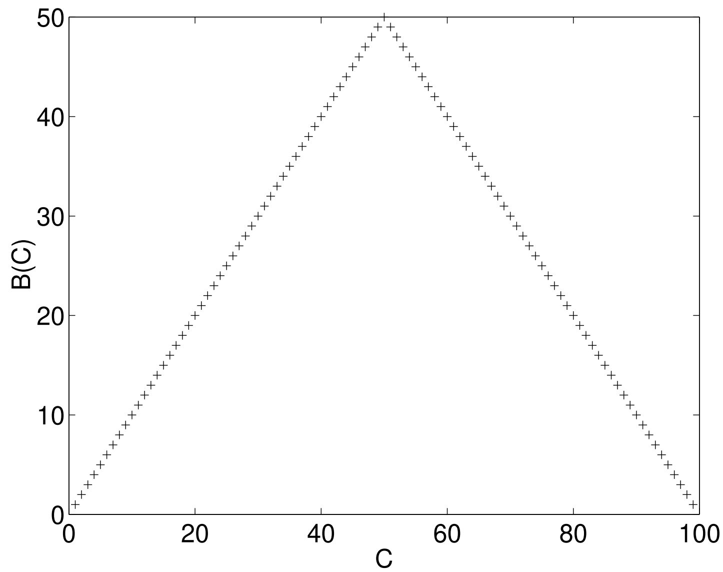

Figure 1: Ultimate betting strategy for $T = 100$.

We refer to the gambler's wealth c_t at any time as the *state* of the game, and the gambler's bet as the *action* to be taken. Thus a strategy B maps states to actions.

We assume for the sake of realism that $p < \frac{1}{2}$, that is, that the game is not fair, with the individual expecting to lose. In [1] Connolly analyzes what he calls the 'ultimate betting strategy.' This strategy requires the bettor to bet as much as is needed to reach the target T or as much as he has, whichever is least. That is, $B(C) = min(T - C, C)\ \forall C$. This betting strategy is illustrated in figure 1 for a target worth of $T = 100$. We use a dynamic programming technique to show that the ultimate strategy is in fact optimal for any T (in terms of maximizing q), but that it is not necessarily unique.

Evaluating Strategies Of interest is the probability q_c of reaching T before going bankrupt given a betting strategy B and capital c. The q_i must satisfy the following relationships.

$$
\begin{aligned}
q_0 &= 0 \\
q_T &= 1 \\
q_i &= pq_{i+B(i)} + (1-p)q_{i-B(i)}. \qquad (1)
\end{aligned}
$$

We can rewrite this as a matrix equation.

$$\begin{aligned} A\vec{q} &= \vec{q} && (2) \\ (A-I)\vec{q} &= \vec{0} && (3) \end{aligned}$$

where

$$\begin{aligned} \vec{q} &= [q_1, q_2, \dots, q_{T-1}, q_T]^T \\ A_{ij} &= \begin{cases} p & i < T,\ i + B(i) = j \\ 1-p & i < T,\ i - B(i) = j \\ 1 & i = j = T \\ 0 & otherwise \end{cases} . \end{aligned}$$

and I is the identity matrix. Thus finding $\vec{q}$ amounts to finding a vector in the null space of $(A - I)$ subject to the condition $q_T = 1$. It can be shown that rank$(A) = T - 1$, so there exists a unique solution.

This gives us a method of evaluating any betting strategy B. We can compute the probability of reaching the target for any initial worth. We illustrate with a simple example with $T = 5$. Let $B(1) = B(4) = 1$ and $B(2) = B(3) = 2$. Then

$$A = \begin{bmatrix} 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & p & 0 \\ 1-p & 0 & 0 & 0 & p \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} .$$

If $p = 9/19$, then $q = [0.173 \;\; 0.365 \;\; 0.565 \;\; 0.771 \;\; 1]^T$, that is, we have almost a 57% chance of reaching our target of 5 if we begin with 3.

Optimal Strategies Once we are able to evaluate a particular strategy, we can find the optimal strategy by a dynamic programming technique. It is important to note that in equation (1) each q_i is a monotonically nondecreasing function of all other q_j. Thus improving the strategy for any particular c cannot decrease q_j for any j. Therefore we can select some value for c, find the bet b that maximizes q_c and update the strategy so that $B(c) = b$. Since there are only finitely many strategies, if we iterate over c, we are guaranteed to converge to the optimal strategy in a finite number of steps.

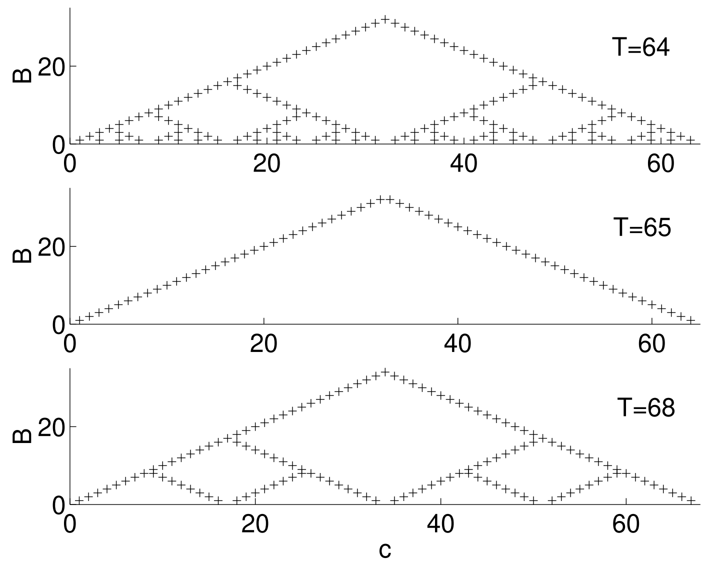

Figure 2: Optimal betting strategies for T = 64,65,68.

2. Expected Losses

Now that we have discovered the optimal wagering strategy, we investigate the gambler's expected losses. We have shown that there may be several optimal betting strategies. Thus a gambler may be able to choose between several different bets to add variety without increasing the risk. We note that Connolly's ultimate strategy is an optimal strategy for any target T, so we will use it for the purposes of the following analysis. Furthermore, we normalize with respect to T and take the limiting case $T \to \infty$, so that C and b are real numbers in $[0,1]$. Then the ultimate strategy is defined by the function $b : [0,1] \to [0,1]$

$$b(c) = \begin{cases} c & 0 \leq c \leq \frac{1}{2} \\ 1-c & \frac{1}{2} \leq c \leq 1 \end{cases}$$

Using the ultimate strategy with probability of success $p = \frac{1}{2} - \delta$, the expected loss given starting capital c obey the following relations. When $0 \leq c \leq \frac{1}{2}$

$$\begin{aligned} L(c) &= (1-p)c - p\ c + p\ L(2c) \\ &= 2\delta\ c + p\ L(2c) \end{aligned}$$

since the loss is c if the first bet is lost, otherwise we have a gain of c and the

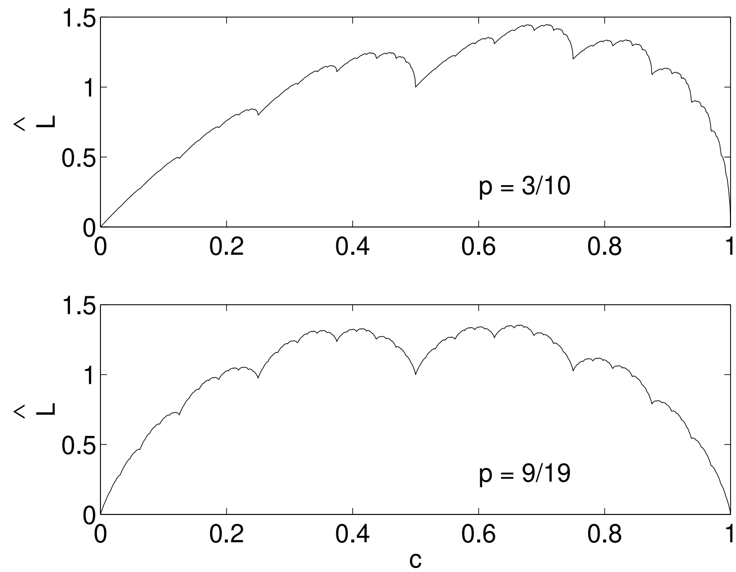

Figure 3: Expected losses for $p = 3/10$ and $p = 9/19$.

expected losses from the new state. When $\frac{1}{2} \leq c \leq 1$

$$
\begin{aligned}
L(c) &= -p(1-c) + (1-p)(1-c) + (1-p)L(2c-1) \\
&= 2\delta\ (1-c) + (1-p)L(2c-1)
\end{aligned}
$$

For the sake of comparison, we normalize with respect to δ, defining

$$\widehat{L}(c) = \frac{L(c)}{\delta}.$$

Then the normalized losses obey the functional relations

$$\widehat{L}(c) = \begin{cases} 0 & c \in \{0,1\} \\ 2c + p\ \widehat{L}(2c) & 0 \leq c \leq \frac{1}{2} \\ 2(1-c) + (1-p)\widehat{L}(2c-1) & \frac{1}{2} \leq c \leq 1 \end{cases} . \tag{7}$$

We show a plot of the normalized expected losses for $p = 3/10 (\delta = 1/5)$ and $p = 9/19$. It is interesting to note that the relation in equation (7) gives rise to a fractal-like curve with apparent self-similarity. In fact, we can write the solution to

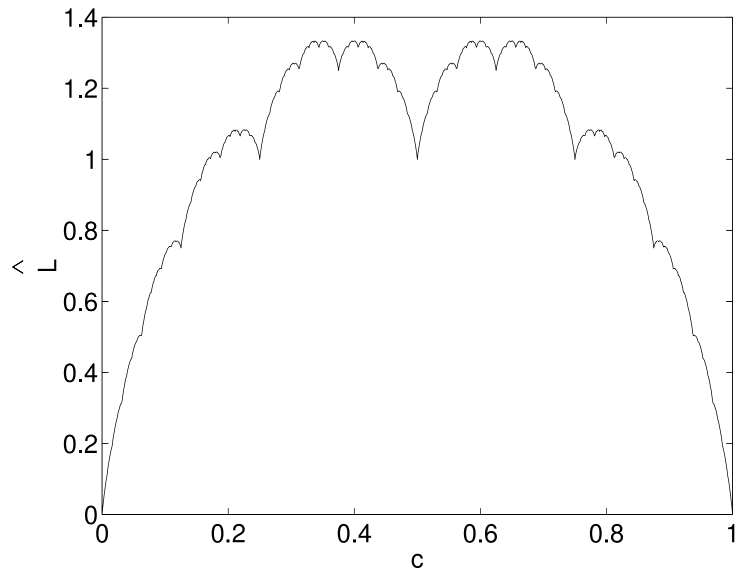

Figure 4: The Knopp function.

(7) as an infinite series.

$$
\begin{aligned}
g_0(x) &= \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2 - 2x & \frac{1}{2} \le x \le 1 \end{cases} \\
g_{i+1}(x) &= \begin{cases} p\ g_i(2x) & 0 \le x \le \frac{1}{2} \\ (1-p)g_i(2x-1) & \frac{1}{2} \le x \le 1 \end{cases} \\
\hat{L}(c) &= \sum_{i=0}^{\infty} g_i(c).
\end{aligned}
$$

$\hat{L}$ is continuous but differentiable nowhere in $[0, 1]$. In the limit as $p \to \frac{1}{2}$, $\hat{L}$ approaches the Knopp function, illustrated in figure 4.

3. Changing The Odds

In many casino games, several different bets may be made, each having its own probability of success and odds paid by the house. Often there are several possible outcomes, and the payoff may take several values. It is straightforward to include these scenarios in the procedure for determining the optimal betting strategy given in section 1. We begin with the simplest extension, that of a single bet with different odds.

Uneven Odds We consider the case in which each game has only 2 outcomes, but the bet pays $M : 1$ odds, with $M \neq 1$. That is, for a bet b, the gambler wins $M \cdot b$ with probability p, and forfeits the bet with probability $1 - p$.

We illustrate the optimal strategy for $M = 2$, but it should be evident that the approach of section 1 can be used by discretizing allowable bets (as is typically the case in casinos) for any rational M.

While the single bet, uneven odds scenario is typical of many sports parimutual wagers, many casino games allow the player to make several different types of bet.

Multiple Betting Options Returning to the game of roulette, we now consider placing any single table bet. These bets are listed in Table 1.

Table 1: Single Roulette bets

$1 : 1$	Even, Odd, Red, Black, 1-18 or 19-36
$2 : 1$	1-12, 13-24, 25-36, 3n, 3n+1 or 3n+2
$8 : 1$	Any four adjacent numbers
$17 : 1$	Any two adjacent numbers
$35 : 1$	Any single number

For any state c, the action is now represented by an ordered pair (b, β), where b is the amount to be bet as before, and β represents which type of bet is to be placed.

Further Extensions We have not addressed scenarios that involve placing multiple simultaneous bets (for example, betting several outcomes at roulette). There are also some cases in which the probabilities or amounts of bets may change before they are settled (for example, splitting at blackjack or placing new bets at craps after the point is established).

4. Conclusion

We have shown that the ultimate betting strategy as described by Connolly is an optimal strategy for achieving a fixed target. This strategy is not necessarily unique in its optimality, though, and we have illustrated the equivalence class of optimal bets. We have also discovered that, using an optimal strategy, the expected loss for a single bet gambling game is a complex and nonintuitive function of the initial wealth.

References

[1] D. Connolly, Casino Gambling, The Ultimate Strategy, *College Math. J.* 30 (1999), 279–278