The Sharpe Ratio, Range and Maximal Drawdown of a Brownian Motion

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Abstract

We analyze some commonly used statistics of a Brownian motion, many of which are important in a financial context, especially as measures of *risk*. Specifically, we consider expected values of the Sharpe ratio (related to the variance), the range and the maximal drawdown, MDD. We develop analytic expressions for the expected value (as well as higher moments) of the extremal points of the Brownian motion, from which the expected value of the range follows. We give an analytic expression for the expected value of the MDD when the drift is zero. For non-zero drift, we provide an infinite series representation, and compute the asymptotics. The MDD exhibits *different* asymptotic behavior for the three cases of zero, positive and negative drift.

1 Introduction

The random walk is fundamental in computational finance, so we discuss some of its properties and the estimation of some of its parameters. Specifically we consider the variance of the brownian motion, the range (the difference between the two extremal values) and the maximal drawdown MDD (the largest peak to bottom drop), all of which are formally defined later. These statistics are commonly used measures for the *risk* of a protfolio whose return rate follows a Brownian motion. The discussion of the variance is not new and is included here for completeness. We have not found explicit analytic expressions for the results presented on the extremal values of the motion (the difference between the two extremal values being the range of the motion), and to the best of our knowledge this is the first serious attempt to study the properties of the MDD.

For a Brownian motion with a drift μ and a variance parameter σ, the *Sharpe ratio* $\mathcal{S}$ is defined by $\mathcal{S} = \mu/\sigma$. Suppose that the Brownian motion is observed in the interval $[0, T]$. For $n > 1$ let $\tau = T/n$. A sample estimate of $\mathcal{S}$ can be obtained by computing the average value and variance of the changes in S over the n time periods of length τ. Calling these average values r_τ and σ_τ^2, the following result holds.

$$E\left[\frac{r_\tau}{\sigma_\tau\sqrt{\tau}}\right] = \mathcal{S}\left(\frac{n}{2}\right)^{1/2}\frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}. \tag{1}$$

As $n \to \infty$, the RHS converges to $\mathcal{S}$. Defining $\alpha = \mathcal{S}\sqrt{T/2}$, we find that the expected value of the range is given by

$$E[R] = \frac{2\sigma}{\mathcal{S}}Q_R(\alpha), \tag{2}$$

where

$$Q_R(x) = erf(x)\left(\frac{1}{2} + x^2\right) + \frac{xe^{-x^2}}{\sqrt{\pi}}. \tag{3}$$

When $\mu = 0$, this formula is accurate in the sense of the limit $\mu \to 0$ ($\mathcal{S} \to 0$), giving

$$E[R] = 2\left(\frac{2\sigma^2 T}{\pi}\right)^{1/2} \qquad \mu = 0. \tag{4}$$

Asymptotically in T, when $\mu \neq 0$, the range grows as $\mu T + \sigma^2/\mu$. For the expected value of the MDD, we find that

$$E[MDD] = \frac{2\sigma}{\mathcal{S}}Q_{MDD}(\alpha^2), \tag{5}$$

where

$$Q_{MDD}(x) = \begin{cases} Q_p(x) & \mu > 0, \\ \gamma\sqrt{2x} & \mu = 0, \\ -Q_n(x) & \mu > 0. \end{cases} \tag{6}$$

$\gamma \approx 0.6226$ is a constant, and, Q_p and Q_n are functions whose exact expressions are quite complicated and are developed later in the text, in sections 2.5.4 and 2.5.5. However, they can be numerically evaluated and their asymptotic behavior can be computed. These results are given in the appendix, section A. The important thing is that the Q functions are universal, not depending on μ, σ or T, and so they only have to be computed once. A graphical illustration of the behavior of Q_p, Q_n and $\gamma\sqrt{2x}$ are shown in Figure 1.

$$Q_p(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ \frac{1}{4}\log x + 0.49088 & x \to \infty. \end{cases} \qquad Q_n(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ x + \frac{1}{2} & x \to \infty. \end{cases} \tag{7}$$

The asymptotic behavior is logarithmic for $\mu > 0$, linear for $\mu < 0$ and square root for $\mu = 0$, quite a divergence in behavior depending on the sign of μ. Infact, this very fact could be used as an hypothesis test for the sign of μ.

In practice, the random walk is only observed at discrete times. Assume the time intervals between observation are a constant, Δt, then the observed MDD will be an underestimate of the true MDD. A correction [Rogers and Satchell, 1991] given by $2\beta_{RS}\sigma\sqrt{\Delta t}$ where $\beta \approx 0.90722$ can be used to augment the observed MDD to get a less biased estimate. A comparison of this correction factor with the true bias is shown in Figure 4.

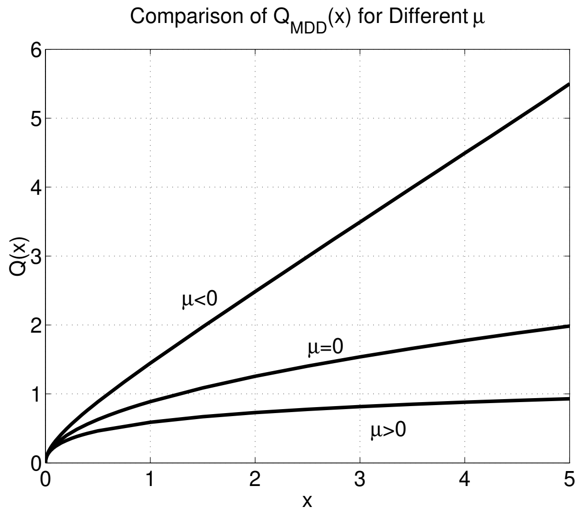

Figure 1: Behavior of the $Q(\cdot)$ functions for positive, negative and zero μ.

The sterling ratio $\mathcal{R}$, a commonly used risk measure, can be defined as $\mathcal{R} = \mu / E\left[MDD\right]$, which we see is then given by

$$\mathcal{R} = \frac{\mathcal{S}^2}{2Q_{MDD}\left(\frac{\mathcal{S}^2 T}{2}\right)}, \tag{8}$$

and thus knowing the Q_{MDD} function and the time period, we have a relationship between the Sharpe ratio and the Sterling ratio. An important note is this relationship is *time dependent*, and thus when the sterling ratio is quoted, it is important to also give the length of time, T over which it was computed – unlike the sharpe ratio, where it is straightforward to normalize out the time dependence, the Sterling ratio is much more complicated, and has quite different properties depending on the sign of μ. The case that is of most practical interest is when $\mu > 0$, in which case Q_{MDD} has asymptotic logarithmic behavior.

1.1 Preliminaries

We begin with the definitions. Let $X(t)$ be the random walk where t is either discrete or continuous time. We are interested in properties of this random walk in the interval $[0, T]$. $X(0) = 0$ and $X(t)$ follows the follows a path given by its stochastic dynamical equation.

Continuous Time. A Standard Brownian motion with drift μ and a variance parameter σ has the following dynamics,

$$dX(t) = \mu dt + \sigma dW(t).$$

Here, $dW(t)$ is a white noise random variable and has the property

$$E\left[\frac{dW(t)}{dt}\right] = 0 \qquad E\left[\frac{dW(t)}{dt}\frac{dW(s)}{ds}\right] = \delta(t-s),$$

where $\delta(\cdot)$ is the dirac delta function. Heuristically $dW(t)$ can be viewed as (any) random variable (usually chosen to be normally distributed) with mean and variance given by $E\left[dW(t)\right] = 0$ and

$E\left[dW(t)^2\right] = dt$. Integrating, one gets

$$X(t) = \mu t + \sigma \int_0^t dW(s) = \mu t + \sigma \int_0^t ds \frac{dW(s)}{ds},$$

which can be used to compute expectations, for example

$$E[X(t)] = \mu t + \sigma \int_0^t ds E\left[\frac{dW(s)}{ds}\right] = \mu T.$$

and

$$E\left[(X(t) - E[X(t)])^2\right] = \sigma^2 \int_0^t ds \int_0^t dv\, \underbrace{E\left[\frac{dW(s)}{ds}\frac{dW(v)}{dv}\right]}_{\delta(s-v)} = \sigma^2 \int_0^t ds = \sigma^2 t,$$

so $Var[X(t)] = \sigma^2 t$.

Discrete Time. Once again, the interval of interest is [0 T], however this interval is broken into n intervals of length $\Delta t = T/n$. The random walk is thus defined on a finite number of points, $X_i = X(i\Delta t)$ for $i = 0, 1, \ldots, n$. We could imagine each step to be Gaussian, however in the limit of small Δt it does not matter, so we assume a finite dynamics for the random walk given by

$$X_{i+1} = \begin{cases} X_i + \delta & \textit{with prob } p, \\ X_i - \delta & \textit{with prob } q = 1 - p. \end{cases}$$

Continuous Time Limit of Discrete Time. Notice that if we want the discrete random walk to simulate the continuous case, then p and δ need to satisfy certain constraints, namely that

$$\begin{aligned} E[X_n - X_0] &= n(p-q)\delta = \mu T, && (9) \\ Var[X_n - X_0] &= \delta^2 = n\sigma^2 \Delta t. && (10) \end{aligned}$$

Solving these two equations for p and δ in terms of the known parameters μ and σ we find that

$$\begin{aligned} \delta &= \sigma\sqrt{\Delta t}\left(1 + \frac{\mu^2 \Delta t}{\sigma^2}\right)^{1/2}, && (11) \\ p &= \frac{1}{2}\left(1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\left(1 + \frac{\mu^2\Delta t}{\sigma^2}\right)^{-1/2}\right), && (12) \\ q &= \frac{1}{2}\left(1 - \frac{\mu\sqrt{\Delta t}}{\sigma}\left(1 + \frac{\mu^2\Delta t}{\sigma^2}\right)^{-1/2}\right). && (13) \end{aligned}$$

Notice that $\mu \to -\mu$ is equivalent to interchanging p with q in the random walk. Notice also that as Δt approaches zero, $p \to \frac{1}{2}$ and the stepsize $\delta \to 0$ both at a rate $\sqrt{\Delta t}$. This is important because in any statistic of the random walk, to get the corresponding statistic for the continuous case, we simply take the limit, letting Δt approach zero, with p and δ having the prescribed dependences. Asymptotically we see that

$$\begin{aligned} \delta &\to \sigma\sqrt{\Delta t}, && (14) \\ p &\to \frac{1}{2}\left(1 + \frac{\mu\sqrt{\Delta t}}{\sigma}\right), && (15) \end{aligned}$$

and the higher order terms can be computed if needed. Exactly such a limiting process to price the European option from first principles [Magdon-Ismail, 2001] and to obtain the joint density of the high and close given the open. Usually the solution of such problems involves the solution of the Fokker-Plank equation and are complex. However, with this limiting process, many problems can be reduced to combinatorics, and taking of the limit, we get the continuous version of the result. It can be shown that the discrete distribution approaches the continuous distribution in the $\Delta t \to 0$ limit. However a statistic that precisely exploits this difference might converge to a different value than its corresponding continuous statistic.

Probability Tools and Notation We will specify the notation that we use throughout. $f_X(x)$ will denote the probability density function for the random variable X evaluated at the point x. We let $F_X(x)$ denote the probability distribution function for the random variable X evaluated at the point x. Note that $f_X(x) = \frac{dF_X(x)}{dx}$ and that $F_X(x) = P[X \leq x]$. We usually assume that $f_X(x)$ is a continuous function of x.

Suppose that X is a non-negative random variable. It is extremely useful to introduce the complementary distribution function $G_X(x) = 1 - F_X(x)$ for the following reason.

Lemma 1.1 *Let $X \geq 0$ be a non-negative random variable and let $q(x)$ be any differentiable non-decreasing function on $[0, \infty)$ for which $E[q(X)] < \infty$. Let $G_X(x) = P[X \geq x]$, be the complementary distribution function. Then,*

$$E[q(X)] = \int_0^\infty dx\ q(x) f_X(x) = q(0) + \int_0^\infty q'(x) G_X(x).$$

Since this is non-standard in probability books, the simple proof is as follows:

$$\begin{aligned}
\int_0^\infty dx\ q(x) f_X(x) &= -\int_0^\infty dx\ q(x)(G_X(x))', && (16)\\
&= -q(x)G_X(x)|_0^\infty + \int_0^\infty dx\ q'(x) G_X(x), && (17)\\
&= q(0) - \lim_{x\to\infty} q(x) G_X(x) + \int_0^\infty dx\ q(x) G_X(x). && (18)
\end{aligned}$$

It thus remains to show that $\lim_{x\to\infty} q(x)G_X(x) = 0$. This follows from the fact that the expectation $E[q(X)]$ exists, which means that $q(x)f_X(x)$ must decay sufficiently fast to make it integrable. This in turn means that $q(x)G_X(x)$ must also decay to zero. More formally, since $E[q(X)]$ exists,

$$\begin{aligned}
0 &= \lim_{A\to\infty} \int_A^\infty dx\ q(x) f_X(x), && (19)\\
&\geq \lim_{A\to\infty} q(A) \int_A^\infty dx\ f_X(x), && (20)\\
&= \lim_{A\to\infty} q(A) G_X(A). && (21)
\end{aligned}$$

The second line follows because $q(\cdot)$ is non-decreasing. Since $q(\cdot)$ is non-decreasing, it cannot go to $-\infty$ so the RHS cannot be less than 0 (as $G_X(A) \to 0$), and hence must equal 0, and the lemma is proved. ■

Using $q(x) = x^k$ we get the moments. Another particularily useful case is $q(x) = e^{ax}$ since in finance, one is often interested in log-transformations. Using the lemma we get that

$$E[X^m] = m \int_0^\infty dx\ x^{m-1} G_X(x), \qquad E[e^{ax}] = 1 + a \int_0^\infty dx\ e^{ax} G_X(x). \tag{22}$$

It turns out that this lemma will be very useful for us because the statistics that we are interested in, for example the maximum of the random walk, the range and the maximum drawdown, are all non-negative. Further, as we shall see, it is possible for us to compute the distribution of these random variables, hence computing whatever expectations we want does not have to go through the added exercise of computing the density.

2 Statistics of the Random Walk

We now consider the statistics of the random walk that we are interested in. Of interest are their expected behaviors as functions of μ, σ and T. Of additional interest are the relationships and correlations among these statistics as it is often the case that one or more of these statistics is available and one might like to infer some information about the others.

2.1 Return

This is simply the value of $X(T)$. A more interesting statistic is r_τ, the τ averaged return defined as follows. Select a time interval τ. We can define the returns over time intervals of length τ by $r_\alpha(\tau) = X(\alpha\tau) - X((\alpha-1)\tau)$. Then r_τ is given by the average of these quantities

$$r_\tau = \frac{1}{n}\sum_{\alpha=1}^{n} r_\alpha(\tau) = \frac{X(n\tau) - X(0)}{n} = \frac{X(T)}{n}, \tag{23}$$

where we assume that $n\tau = T$. As already discussed, $r_\alpha(\tau)$ is distributed according to a Gaussian with mean $\mu\tau$ and variance $\sigma^2\tau$, and since the $r_\alpha(\tau)$ are independent, r_τ will have a distribution with the same mean and a variance decreased by a factor of $1/n$. Thus,

$$r_\tau \sim N(\mu\tau, \sigma^2\tau/n), \tag{24}$$

where we use $N(\mu, \sigma^2)$ to denote a Gaussian distribution with mean μ and variance σ^2. Remember that $X(T) \sim N(\mu T, \sigma^2 T)$.

2.2 Variance

This is simply the variance of $X(T)$. Since this is not observable, a more interesting statistic is σ_τ^2, the τ-variance. defined by the variance of the $r_\alpha(\tau)$'s.

$$\sigma_\tau^2 \;=\; \frac{1}{n}\sum_{\alpha=1}^{n}(r_\alpha(\tau) - r_\tau)^2 = \frac{1}{n}\sum_{\alpha=1}^{n} r_\alpha^2(\tau) - r_\tau^2 = \frac{1}{n}\sum_{\alpha=1}^{n} r_\alpha^2(\tau) - \frac{X^2(T)}{n^2}. \tag{25}$$

The following theorem is useful

Theorem 2.1 (see for example [DeGroot, 1989]) *Suppose that $X_1, \ldots, X_n$ form a random, i.i.d sample from a normal distribution with mean μ and variance σ^2. Then the sample mean, $\bar{X}_n = (1/n)\sum_{i=1}^{n} X_i$ and the sample variance $\hat{\sigma}^2 = (1/n)\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$ are independent random variables; $\bar{X}_n$ has a normal distribution with mean μ and variance σ^2/n and $n\hat{\sigma}^2/\sigma^2$ has a χ^2 distribution with $n-1$ degrees of freedom (χ^2_{n-1}).*

Thus, in a Gaussian random sample, the variance and the mean are statistically independent – this is intuitive because knowing the mean places no restrictions whatsoever on the variance, however that it should only hold for Gaussian random numbers is not intuitive, but true. Thus, r_τ and σ_τ^2 are independent statistics of the random walk. Further, $n\sigma_\tau^2/\sigma^2\tau$ has a χ^2_{n-1} distribution - a type of gamma distribution. Thus we can immediately write down the density for σ_τ^2 using the formula that if $X \sim f_X(x)$ and if $Z = aX$ with density $f_Z(z)$ then $f_Z(z) = \frac{1}{|a|}f_X(z/a)$. We get that

$$f_{\sigma_\tau^2}(s) = \left(\frac{n}{\sigma^2\tau}\right)^{(n-1)/2} \frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)} s^{(n-1)/2-1} e^{-ns/2\sigma^2\tau}. \tag{26}$$

A useful formula is the expected value of σ_τ^K.

$$E\left[\sigma_\tau^K\right] = \left(\frac{2\sigma^2\tau}{n}\right)^{K/2} \frac{\Gamma(\frac{n-1}{2}+\frac{K}{2})}{\Gamma(\frac{n-1}{2})} \qquad K > -(n-1). \tag{27}$$

Thus, for example, one could compute the mean and variance of σ_τ^2.

$$E\left[\sigma_\tau^2\right] = \frac{n-1}{n}\sigma^2\tau, \tag{28}$$

and is thus not quite an unbiased estimate of $\sigma^2\tau$. The variance is given by

$$Var\left[\sigma_\tau^2\right] = \frac{2(n-1)}{n^2}\sigma^4\tau^2 \sim \frac{1}{n}. \tag{29}$$

Note also that since we know the distributions of r_τ and σ_τ^2, and that they are independent, we can write down the full joint distribution, $f_{r_\tau,\sigma_\tau^2}(r,s)$ by simply taking the product of the two respective distributions.

2.3 Sharpe Ratio

The ratio of the return to the standard deviation is important in financial circles. Notice that $E\left[r_\tau\right] = \mu\tau$ and $E\left[\sigma_\tau^2\right] = \sigma^2\tau$ so we do not expect the ratio r_τ/σ_τ to yield a quantity that is independent of τ, which would be desirable as τ has up to now been somewhat arbitrary. In order to obtain a τ–independent statistic, we thus define the sample Sharpe ratio $\hat{\mathcal{S}}$ by

$$\hat{\mathcal{S}} = \frac{r_\tau}{\sigma_\tau\sqrt{\tau}}. \tag{30}$$

Since r_τ and σ_τ are independent, we can get the expected value of $\hat{\mathcal{S}}$ by taking the product of the expectations. Using (27) with $K = -1$, this yields

$$E\left[\hat{\mathcal{S}}\right] = \frac{\mu}{\sigma}\left(\frac{n}{2}\right)^{1/2} \frac{\Gamma(\frac{n}{2}-1)}{\Gamma(\frac{n}{2}-\frac{1}{2})}. \tag{31}$$

Note that as $n \to \infty$, $E\left[\hat{\mathcal{S}}\right] \to \mathcal{S}$. In fact, it is possible to get the full distribution of the sample Sharpe ratio by taking the product of the distribution of r_τ and the distribution of $1/\sigma_\tau$. The result is

$$f_{\hat{\mathcal{S}}}(s) = \frac{2e^{-\frac{\mu^2}{2\sigma^2}\left(\frac{s^2}{\sigma^2}-(n-1)\tau\right)}\left(1+\frac{s^2}{\sigma^2(n-1)\tau}\right)^{-\frac{n}{2}}}{2^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\sqrt{2\pi\sigma^2(n-1)\tau}} \int_0^\infty dx\; x^{n-1} e^{-\frac{1}{2}\left(x-\frac{\mu s}{\sigma^2}\right)}. \tag{32}$$

When $\mu = 0$ this becomes a t distribution with $n-1$ degrees of freedom. Since $\sqrt{n}(r_\tau - \mu\tau)/\sigma\sqrt{\tau}$ is standard normal, and $n\sigma_\tau^2/\sigma^2\tau$ is χ^2_{n-1}, the ratio,

$$\mathcal{C} = \frac{\dfrac{\sqrt{n}(r_\tau - \mu\tau)}{\sigma\sqrt{\tau}}}{\left(\dfrac{n\sigma_\tau^2}{(n-1)\sigma^2\tau}\right)^{1/2}} = \frac{\sqrt{n-1}(r_\tau - \mu\tau)}{\sigma_\tau}, \tag{33}$$

has by definition a t distribution with $n-1$ degrees of freedom, which is independent of σ^2! Further, note that $\mathcal{C}$ itself is independent of σ^2 and is related to the sample Sharpe ratio $\hat{\mathcal{S}}$.

2.4 High, Low and Range

The high H, low L and the range R are defined as expected.

$$H = \sup_{t\in[0,T]} X(t), \qquad L = \inf_{t\in[0,T]} X(t), \qquad R = H - L. \tag{34}$$

We will derive the exact value of the expectations of these quantities as well as the distribution of the H and L. It is also possible to get the joint distribution of H and L [Magdon-Ismail and Atiya, 2000] and hence the distribution of R, but we postpone this tedious computation. Additionally, the distribution of the cover time for a brownian motion can be found in [Chong *et al.*, 1999, Imhof, 1985], from which after a differentiation, one can obtain the distribution for the range. It suffices to consider the distribution of H since the distribution of $|L|$ can be obtained from the distribution of H by setting $\mu \to -\mu$. We will compute the expected value of the range, the high and the low using the distribution of the high. We have not been able to find this explicit result in the literature for the general asymetric brownian motion, though for the symmetric case, [Feller, 1951] gives a result.

Consider a random walk with an absorbing barrier at h and let λ be the time at which the random walk gets absorbed. The distribution $f_\lambda(t)$ is known (see for example [Dominé, 1996]) and is given by the inverse Gaussian distribution

$$f_\lambda(t) = \frac{h}{(2\pi\sigma^2 t^3)^{1/2}} e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{35}$$

Thus, the probability that $H \geq h$ is exactly the probability that the random walk gets absorbed in the interval $[0, T]$. Hence, using the fact that $G_H(h) = \int_0^T dt\ f_\lambda(t)$, we have

$$G_H(h) = h\int_0^T \frac{dt}{t}\ \frac{1}{(2\pi\sigma^2 t)^{1/2}} e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{36}$$

Using Lemma 2.1, we can now compute the moments of H as as follows.

$$E\left[H^m\right] = m\int_0^\infty dh\ h^{m-1} G_H(h), \tag{37}$$

$$= m\int_0^T \frac{dt}{t}\ \frac{1}{(2\pi\sigma^2 t)^{1/2}} \int_0^\infty dh\ h^m e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{38}$$

We illustrate how to use this formula to obtain the moments explicitly for small m. The general case can be evaluated as a finite series if so desired.

E [H]: For the case $m = 1$ we have the expected value of the high. Making a change of variables in the h–integral to $u = (h - \mu t)/(2\sigma^2 t)^{1/2}$ we find that

$$
\begin{aligned}
E\left[H\right] &= \frac{1}{\sqrt{\pi}} \int_0^T dt \left[\int_{-\alpha(t)}^{\infty} du \, \left(\frac{2\sigma^2}{t} \right)^{1/2} u e^{-u^2} + \mu \int_{-\alpha(t)}^{\infty} du \; e^{-u^2} \right], && (39) \\
&= \frac{1}{\sqrt{\pi}} \int_0^T dt \left(\frac{\sigma^2}{2t} \right)^{1/2} e^{-\alpha^2(t)} + \frac{\mu T}{2} + \frac{\mu}{\sqrt{\pi}} \int_0^T dt \int_0^{\alpha(t)} du \; e^{-u^2}, && (40)
\end{aligned}
$$

where $\alpha(t) = \mu t^{1/2}/(2\sigma^2)^{1/2}$. Defining the error function,

$$
erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x du \; e^{-u^2}, \tag{41}
$$

the first integral (after a change of variables) can be reduced to $\frac{\sigma^2}{\mu} erf(\alpha(T))$. In order to do the second integral, we use a change of variable from t to $x = \alpha(t)$. The last integral then becomes

$$
\frac{4\sigma^2}{\mu\sqrt{\pi}} \int_0^{\alpha(T)} dx \; x \int_0^x du \; e^{-u^2}.
$$

Integrating by parts on $x \int_0^x du \; e^{-u^2}$ we find that

$$
\int_0^{\alpha(T)} dx \; x \int_0^x du \; e^{-u^2} = \frac{\sqrt{\pi}}{4} erf(\alpha(T)) \left(\alpha^2(T) - \frac{1}{2} \right) + \frac{\alpha(T)}{4} e^{-\alpha^2(T)},
$$

and thus we finally get for the expectation

$$
E\left[H\right] = \frac{\mu T}{2} + \frac{\sigma^2}{\mu} \left[erf(\alpha) \left(\frac{1}{2} + \alpha^2 \right) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}} \right], \tag{42}
$$

where by α we mean $\alpha(T)$. Expectations for the low and the range can now also be derived.

$$
\begin{aligned}
E\left[L\right] &= -E\left[H|-\mu\right] = -\frac{\mu T}{2} + \frac{\sigma^2}{\mu} \left[erf(\alpha) \left(\frac{1}{2} + \alpha^2 \right) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}} \right], && (43) \\
E\left[R\right] &= E\left[H|\mu\right] + E\left[H|-\mu\right] = \frac{2\sigma^2}{\mu} \left[erf(\alpha) \left(\frac{1}{2} + \alpha^2 \right) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}} \right]. && (44)
\end{aligned}
$$

Defining $Q_R(\alpha)$ by

$$
Q_R(\alpha) = erf(\alpha) \left(\frac{1}{2} + \alpha^2 \right) + \frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}}, \tag{45}
$$

these formulae can be written more compactly as

$$
E\left[H\right] = \frac{\sigma^2}{\mu} \Big(\alpha^2 + Q_R(\alpha) \Big), \qquad E\left[L\right] = \frac{\sigma^2}{\mu} \Big(\alpha^2 - Q_R(\alpha) \Big), \qquad E\left[R\right] = \frac{2\sigma^2}{\mu} Q_R(\alpha). \tag{46}
$$

We now consider several interesting limits of these formulae. We use the following asymptotic relationships for $erf(x)$, [Gradshteyn and Ryzhik, 1980]

$$
erf(x) \to \begin{cases} \dfrac{2}{\sqrt{\pi}} \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)(k-1)!} & x \to 0, \\ 1 - \dfrac{e^{-x^2}}{\pi} \displaystyle\sum_{k=0}^{n-1} \frac{(-1)^k \Gamma(k+\frac{1}{2})}{x^{k+\frac{1}{2}}} + R_n & x \to \infty. \end{cases} \tag{47}
$$

When $\mu = 0$, we find that

$$E[R] = 2\left(\frac{2\sigma^2 T}{\pi}\right)^{1/2} \qquad \mu = 0. \tag{48}$$

The behavior as $T \to 0$ and $T \to \infty$ are given (for $\mu \geq 0$) by[1]

$$E[R] = \begin{cases} \dfrac{2\sigma^2}{\mu}\left(\dfrac{2\alpha}{\sqrt{\pi}} + \dfrac{2\alpha^3}{\sqrt{\pi}} + \cdots\right) & \alpha \to 0, \\ \dfrac{2\sigma^2}{\mu}\left(\alpha^2 + \dfrac{1}{2} - \dfrac{e^{-\alpha^2}}{\alpha^3} + \cdots\right) & \alpha \to \infty. \end{cases} \tag{49}$$

Thus two different kinds of behavior emerge at the different limits. This is in contrast to the Sharpe ratio which is roughly constant, independent of T, as can be seen from the expected value of the sample Sharpe ratio (31) which only depends on n (which is a function of $\tau = T/n$). Further, $(n/2)^{1/2}\Gamma(n/2-1)/\Gamma(n/2-1/2)$ approaches 1 for large n, and is fairly robust with respect to the choice of τ.

We can use similar methods to obtain $E\left[H^2\right]$. From (38), we get that

$$E\left[H^2\right] = 2\int_0^T \frac{dt}{t}\frac{1}{(2\pi\sigma^2 t)^{1/2}}\int_0^\infty dh\; h^2 e^{-\frac{(h-\mu t)^2}{2\sigma^2 t}}. \tag{50}$$

Once again, making a change of variables to $u = (h-\mu t)/\sqrt{2\sigma^2 t}$ in the h integral, we arrive at

$$E\left[H^2\right] = \frac{4}{\sqrt{\pi}}\int_0^T dt \int_{-\alpha(t)}^\infty du\;\left(\sigma^2 u^2 e^{-u^2} + (2\sigma^2 t)^{1/2}\mu u e^{-u^2} + \frac{\mu^2 t}{2}e^{-u^2}\right), \tag{51}$$

which means we need to do three integrals. Making a change of variables to $x = \alpha(t)$ in the t integral, and then integrating by parts where necessary, each of these three integrals can be converted into an intagral of the form

$$I_k(\alpha) = \int_0^{\alpha(t)} dx\; x^k e^{-x^2}, \tag{52}$$

where $\alpha = \alpha(T)$. The following relationships allow us to complete all these integrals.

$$I_0(\alpha) = \frac{\sqrt{\pi}}{2} erf(\alpha), \tag{53}$$

$$I_1(\alpha) = \frac{1}{2}\left(1 - e^{-\alpha^2}\right), \tag{54}$$

$$I_k(\alpha) = \frac{k-1}{2} I_{k-2}(\alpha) - \frac{\alpha^{k-1}e^{-\alpha^2}}{2} \qquad k > 1. \tag{55}$$

The final result is

$$E\left[H^2|\mu,\sigma\right] = \frac{\sigma^4}{\mu^2}\left[\frac{\alpha e^{-\alpha^2}}{\sqrt{\pi}} - \frac{erf(\alpha)}{2} + 2(\alpha^2+\alpha^4)(1+erf(\alpha)) + \frac{2\alpha^3 e^{-\alpha^2}}{\sqrt{\pi}}\right]. \tag{56}$$

We obtain $E\left[L^2\right]$ by changing μ to $-\mu$, i.e., $E\left[L^2|\mu,\sigma\right] = E\left[H^2|-\mu,\sigma\right]$. We can get the variance by using the identity $Var[H] = E\left[H^2\right] - E\left[H\right]^2$.

[1] Note that $E[R|\mu] = E[R|-\mu]$.

2.5 Maximum Draw Down (MDD)

Loosely speaking, this is the largest drop from a peak to a low. Unlike the range which is simply the max minus the min, for the MDD, the order in which the max and min occur matters. If the min occurs after the max, then $MDD = R$. In general, there appears to be no straightforward relationship between R and MDD (except that $R \geq MDD$). We will investigate the behavior of MDD here. A simple linear time algorithm for computing MDD on a discrete realization of the Brownian motion is as follows. Assume that $X(t)$ is observed at the $N+1$ times $t_0, t_1, \ldots, t_N$.

```
CURMAX=t0;D=0;MDD=0; // D=Current Drawdown
for i=1 to N{
    if (X(ti)>CURMAX) then CURMAX=X(ti);
    D=CURMAX-X(ti);
    if (D>MDD) then MDD=D;
}
```

Formally we can define MDD as

$$MDD = \sup_{t\in[0,T]}\left[\sup_{s\in[0,t]} X(s) - X(t)\right]. \tag{57}$$

2.5.1 Discrete Random Walk

Let D_t be the draw down from the previous max at time t, $D_0 = 0$. D_t is a random walk that has dynamics very similar to X_t. If X_t goes down (with probability $q = 1-p$) then D_t goes up. If X_t goes up (with probability p) then D_t goes down, with the exception that D_t cannot drop below 0. In otherwords, if X_t follows a random walk with probability p, then D_t follows a random walk with probability $1-p$ and a reflecting barrier at 0, which is where the random walk starts. MDD is then given by

$$MDD = \max_t D_t. \tag{58}$$

If we add an absorbing barrier at h, the random walk gets absorbed if $D_t \geq h$ for any time in the interval of interest. Thus we can get the probability that $MDD > h$ by considering the absorption probability for this random walk. Let $f(i|h)$ be the probability that the random walk gets absorbed at exactly time step i. Then

$$P[MDD > h] = P[\text{absorbtion} \in [0,T]] = \sum_{i=0}^{T/\Delta t} f(i|h). \tag{59}$$

$f(i|h)$ was initially computed in [Weesakul, 1961] for $p/q < (1+1/N)^2$, the more general case being given in [Blasi, 1976], which after the correction of some typographic errors is given by

$$f(i|h) = \begin{cases} \tilde{f}(1) & \frac{p}{q} < \left(1+\frac{1}{N}\right)^2, \\ \tilde{f}(2) + \frac{3}{2}\frac{2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)}}{(N+1)(N+\frac{1}{2})} & \frac{p}{q} = \left(1+\frac{1}{N}\right)^2, \\ \tilde{f}(2) + \frac{2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} q^{\frac{1}{2}} \cosh^{i-1}\beta \sinh^2\beta}{(N+1)q^{\frac{1}{2}}\cosh(N+1)\beta - Np^{\frac{1}{2}}\cosh N\beta} & \frac{p}{q} > \left(1+\frac{1}{N}\right)^2, \end{cases} \tag{60}$$

where $N = h/\delta$, and

$$\tilde{f}(k) = -2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} \sum_{v=k}^{N} \frac{q^{\frac{1}{2}} \cos^{i-1}\alpha_v \sin^2\alpha_v}{(N+1)q^{\frac{1}{2}}\cos(N+1)\alpha_v - Np^{\frac{1}{2}}\cos N\alpha_v}, \tag{61}$$

and where, $\alpha_v \in \left(\frac{v\pi}{N-1}, \frac{(v+1)\pi}{N-1}\right)$ satisfies

$$q^{\frac{1}{2}}\sin(N+1)\alpha_v - p^{\frac{1}{2}}\sin N\alpha_v = 0, \tag{62}$$

and β satisfies

$$q^{\frac{1}{2}}\sinh(N+1)\beta - p^{\frac{1}{2}}\sinh N\beta = 0. \tag{63}$$

Note that δ and p are given in (13). Taking the limit $\Delta t \to 0$, should give the continuous case. Suppose that we let $\hat{f}_\tau(t|h)$ be the corresponding density for absorbtion in the time interval $[t, t+\Delta t]$ defined by $\hat{f}_\tau(t|h)\Delta t = f(t/\Delta t|h)$. Hence

$$P[MDD > h] = \sum_{i=0}^{T/\Delta t} \Delta t \hat{f}_\tau(i\Delta t|h), \tag{64}$$

which is the Rieman sum approximation to an integral. Since as $\Delta t \to 0$, $\hat{f}_\tau(t|h) \to f_\tau(t|h)$, the continuous time absorbtion density, we have that in the $\Delta t \to 0$ limit,

$$P[MDD > h] = \int_0^T dt\ f_\tau(t|h). \tag{65}$$

It thus remains to take the limit $\Delta t \to 0$ of $\hat{f}_\tau(t|h) = f(i|h)/\Delta t$. In this limit, $p = \frac{1}{2}(1+\lambda)$ and $q = \frac{1}{2}(1-\lambda)$ there $\lambda \to \frac{\mu\sqrt{\Delta t}}{\sigma}$ and $\delta \to \sigma\sqrt{\Delta t}$. Since $p/q \to 1 + 2\mu\sqrt{\Delta t}/\sigma$ and $(1+1/N)^2 \to 1 + 2\delta/h \to 1 + 2\sigma\sqrt{\Delta t}/h$, the three cases in (60) are given by $\mu < \sigma^2/h$, $\mu = \sigma^2/h$, and $\mu > \sigma^2/h$. We also find that

$$\begin{aligned} 2^i p^{\frac{1}{2}(i-N)} q^{\frac{1}{2}(i+N)} &= (1-\lambda^2)^{\frac{i}{2}} \left(\frac{1-\lambda}{1+\lambda}\right)^{\frac{N}{2}}, & (66)\\ &\to \left(1 - \frac{\mu^2\Delta t}{\sigma^2}\right)^{\frac{t}{2\Delta t}} \left(1 - \frac{2\mu\sqrt{\Delta t}}{\sigma}\right)^{\frac{h}{2\sigma\sqrt{\Delta t}}}, & (67)\\ &\to e^{-\frac{\mu^2 t}{2\sigma^2}} e^{-\frac{\mu h}{\sigma^2}}. & (68) \end{aligned}$$

We now look at the eigenvalue condition on α_v.

$$q^{\frac{1}{2}}\sin(N+1)\alpha_v - p^{\frac{1}{2}}\sin N\alpha_v = 0 \quad \Rightarrow \quad (1-\lambda)^{\frac{1}{2}}\sin(N+1)\alpha_v - (1+\lambda)^{\frac{1}{2}}\sin N\alpha_v = 0.$$

Since $\lambda \to 0$, we take the first order expansion in λ to get

$$\sin(N+1)\alpha_v - \sin N\alpha_v = \frac{\lambda}{2}(\sin(N+1)\alpha_v + \sin N\alpha_v). \tag{69}$$

The following identities are useful

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}, \tag{70}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}, \tag{71}$$

finally giving the condition

$$\tan\left(N+\frac{1}{2}\right)\alpha_v \cos\alpha_v = \frac{2}{\lambda}\sin\frac{\alpha_v}{2}. \tag{72}$$

Letting $\theta_v = (N+1/2)\alpha_v$, and noting that for every fixed v, $\alpha_v \in \left(\frac{v\pi}{N-1}, \frac{(v+1)\pi}{N-1}\right) \to 0$, we can take the limit of this eigenvalue equation (remembering that $N\lambda \to \mu h/\sigma^2$) to get

$$\tan\theta_v = \frac{\sigma^2}{\mu h}\theta_v, \tag{73}$$

with $\theta_v \in \left(v\pi\frac{N+\frac{1}{2}}{N-1}, (v+1)\pi\frac{N+\frac{1}{2}}{N-1}\right) \to (v\pi, (v+1)\pi]$. In an identical manner, we can analyse the eigenvalue condition for β to get

$$\tanh\left(N+\frac{1}{2}\right)\beta\cosh\beta = \frac{2}{\lambda}\sinh\frac{\beta}{2}. \tag{74}$$

Defining $\eta = (N+\frac{1}{2})\beta$, and taking the limit, we find

$$\tanh\eta = \frac{\sigma^2}{\mu h}\eta. \tag{75}$$

We now take the limit of the summand in $\tilde{f}$, which we first rewrite using the fact that $\sin\alpha_v \to \theta_v/(N+\frac{1}{2})$ as

$$\frac{\theta_v^2\cos^{i-1}\alpha_v}{(N+\frac{1}{2})^2(N+1)\left[\cos(\theta_v+\frac{1}{2}\alpha_v) - A\cos(\theta_v-\frac{1}{2}\alpha_v)\right]}, \tag{76}$$

where

$$A = \frac{(1+\lambda)^{1/2}}{(1+\frac{1}{N})(1-\lambda)^{1/2}}.$$

Using the fact that $\lim_{x\to 0}\cos^{1/x^2}x = e^{-1/2}$, we can analyse $\cos^{i-1}\alpha_v$. Since $i = t/\Delta t$, $\cos^{i-1}\alpha_v = cos^{t/\Delta t-1}\alpha_v \to cos^{t/\Delta t}\alpha_v$. Since $N\to\infty$, we see that $\alpha_v = \theta_v/(N+\frac{1}{2}) \to \sigma\theta_v\sqrt{\Delta t}/h$ hence $t/\Delta t \to \sigma^2\theta_v^2 t/h^2\alpha_v^2$. Thus, we get that

$$\cos^{i-1}\alpha_v \to e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}. \tag{77}$$

We can neglect constants with respect to N, and $1/N^2 = \sigma^2\Delta t/h^2$. Thus, using the double angle formulae, we get for the summand,

$$\frac{\Delta t\sigma^2}{h^2}\frac{\theta_v^2 e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}}{N(1-A)\cos\theta_v\cos\frac{1}{2}\alpha_v - N(1+A)\sin\theta_v\sin\frac{1}{2}\alpha_v}. \tag{78}$$

In the limit,

$$N(1+A)\sin\frac{1}{2}\alpha_v \quad\to\quad \theta_v \qquad (\text{since } A\to 1), \tag{79}$$

$$N(1-A)\cos\frac{1}{2}\alpha_v \quad\to\quad 1-N\lambda \to 1-\mu h/\sigma^2. \tag{80}$$

Thus we finally get for $\tilde{f}$

$$\frac{\Delta t\sigma^2}{h^2}\frac{\theta_v^2 e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}}{(1-\frac{\mu h}{\sigma^2})\cos\theta_v - \theta_v\sin\theta_v}. \tag{81}$$

The following manipulations are useful.

$$\begin{aligned}
\frac{\theta_v^2}{(1-\frac{\mu h}{\sigma^2})\cos\theta_v - \theta_v\sin\theta_v} &= \frac{-\sigma^4\theta_v^3}{\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]} && \text{from (73)},\\
&= \frac{-\sigma^4\theta_v^3\sin\theta_v}{\sin^2\theta_v[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]},\\
&= \frac{-\sigma^4\theta_v^3\sin\theta_v[1+\tan^2\theta_v]}{\tan^2\theta_v[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]} && \text{since } \sin^2 x = \frac{\tan^2 x}{1+\tan^2 x},\\
&= \frac{-\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]}{[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]} && \text{from (73)}.
\end{aligned} \tag{82}$$

Plugging all these results back into (60) and dividing by Δt, we finally arrive at the continuous limit of $\tilde{f}$,

$$\tilde{f}(k) = e^{-\frac{\mu^2 t}{2\sigma^2}} e^{-\frac{\mu h}{\sigma^2}} \frac{\sigma^2}{h^2}\sum_{v=k}^{\infty}\frac{\theta_v\sin\theta_v[\sigma^4\theta_v^2+\mu^2h^2]e^{-\frac{\sigma^2\theta_v^2 t}{2h^2}}}{[\sigma^4\theta_v^2+\mu^2h^2-\mu h\sigma^2]}, \tag{83}$$

where θ_v are the positive solutions to the eigenvalue condition $\tan\theta_v = \sigma^2\theta_v/\mu h$. For the second case, the additional term can be computed using (68), the fact that $\mu h/\sigma^2 = 1$ and that $1/N^2 \to \sigma^2\Delta t/h^2$. On dividing by Δt we get for the additional term,

$$e^{-\frac{\mu^2 t}{2\sigma^2}}\frac{3\sigma^2}{2eh^2} \qquad\qquad \mu = \frac{\sigma^2}{h}. \tag{84}$$

For the third case, we use techniques similar to those used in getting the summand of $\tilde{f}$. We have that $\beta = \eta/(N+\frac{1}{2}) \to 0$, and $\lim_{x\to 0}\cosh^{1/x^2} x = e^{1/2}$, so we see that $\cosh^{i-1}\beta \to \exp(\frac{\sigma^2\eta^2 t}{2h^2})$. Thus, as with (78), we get for the additional term in the third case,

$$\frac{\Delta t\sigma^2}{h^2}\frac{\eta^2 e^{\frac{\sigma^2\eta^2 t}{2h^2}}}{N(1-A)\cosh\eta\cosh\frac{1}{2}\beta - N(1+A)\sinh\eta\sinh\frac{1}{2}\beta}, \tag{85}$$

which upon using manipulations similar to those that led to (81) and (82), we finally arrive at

$$\frac{\eta\sinh\eta(\mu^2h^2-\sigma^4\eta^2)}{(\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h)}. \tag{86}$$

Using (68) and dividing by Δt, the additional term in the third case becomes

$$\frac{\sigma^2}{h^2}\frac{(\mu^2h^2-\sigma^4\eta^2)\eta\sinh\eta}{(\sigma^4\eta^2-\mu^2h^2+\sigma^2\mu h)}e^{-\frac{\mu^2 t}{2\sigma^2}}e^{-\frac{\mu h}{\sigma^2}}e^{\frac{\sigma^2\eta^2 t}{2h^2}} \qquad\qquad \mu > \frac{\sigma^2}{h}. \tag{87}$$

Using (83), (84) and (87) in (60), we arrive at the continuous limit of the discrete time density.

2.5.2 Continuous Time Directly

We reformulate the problem in continuous time in an identical way. Namely, suppose that $X(t)$ is the Brownian motion and that $D(t)$ is the corresponding Brownian motion for the draw down. As with the discrete case, if $X(t)$ has drift and variance parameters μ_B and σ^2, then $D(t)$ follows a Brownian motion with drift $\mu_D = -\mu_B$ and variance parameter σ^2. $D(0) = 0$ and the process has a reflective barrier at 0. Let there be an absorbing barrier at h and let $f_\tau(t|h)$ be the probability density for being absorbed at t. Let $G(h|T)$ be the probability that $MDD \geq h$ in the interval $[0, T]$. Then,

$$G(h|T) = \int_0^T dt\ f_\tau(t|h). \tag{88}$$

Given the complementary distribution function, we can get the expected value of MDD from Lemma 1.1 as $E\,[MDD|T] = \int_0^\infty dh\ G(h|T)$. The absorbtion time density for $D(t)$ when the Brownian has mean $\mu_B = \mu$ has been computed in [Dominé, 1996] and is given by

$$f_\tau(t|h) = e^{-\frac{\mu^2 t}{2\sigma^2}} \left[\frac{\sigma^2}{h^2} \sum_{n=0}^{\infty} \frac{(\sigma^4\theta_n^2 + \mu^2 h^2)\theta_n \sin\theta_n}{(\sigma^4\theta_n^2 + \mu^2 h^2 - \sigma^2\mu h)} e^{-\frac{\mu h}{\sigma^2}} e^{-\frac{\sigma^2\theta_n^2 t}{2h^2}} + K \right], \tag{89}$$

where θ_n are the positive solutions to the eigenvalue condition

$$\tan\theta_n = \frac{\sigma^2}{\mu h}\theta_n, \qquad \theta_n \in (n\pi, (n+1)\pi)\,, \tag{90}$$

and K has the form

$$K = \begin{cases} 0 & \mu < \dfrac{\sigma^2}{h}, \\ \dfrac{3\sigma^2}{2eh^2} & \mu = \dfrac{\sigma^2}{h}, \\ \dfrac{\sigma^2}{h^2} \dfrac{(\mu^2 h^2 - \sigma^4\eta^2)\eta \sinh\eta}{(\sigma^4\eta^2 - \mu^2 h^2 + \sigma^2\mu h)} e^{-\frac{\mu h}{\sigma^2}} e^{\frac{\sigma^2\eta^2 t}{2h^2}} & \mu > \dfrac{\sigma^2}{h}, \end{cases} \tag{91}$$

where η is the positive solution to the eigenvalue condition

$$\tanh\eta = \frac{\sigma^2}{\mu h}\eta. \tag{92}$$

We can see that this solution agrees with the continuous limit of the discrete process, from (83), (84) and (87). Since $G_{MDD}(h|T) = \int_0^T dt\ f_\tau(t|h)$, we find that

$$G_{MDD}(h|T) = 2\sigma^4 \sum_{n=1}^{\infty} \frac{\theta_n \sin\theta_n}{(\sigma^4\theta_n^2 + \mu^2 h^2 - \sigma^2\mu h)} e^{-\frac{\mu h}{\sigma^2}} \left(1 - e^{-\frac{\sigma^2\theta_n^2 T}{2h^2}} e^{-\frac{\mu^2 T}{2\sigma^2}}\right) + L, \tag{93}$$

where L is given by

$$L = \begin{cases} 0 & \mu < \dfrac{\sigma^2}{h}, \\ \dfrac{3}{e}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2}}\right) & \mu = \dfrac{\sigma^2}{h}, \\ \dfrac{2\sigma^4\eta \sinh\eta\ e^{-\frac{\mu h}{\sigma^2}}}{(\sigma^4\eta^2 - \mu^2 h^2 + \sigma^2\mu h)} \left(1 - e^{-\frac{\mu^2 T}{2\sigma^2}} e^{\frac{\sigma^2\eta^2 T}{2h^2}}\right) & \mu > \dfrac{\sigma^2}{h}. \end{cases} \tag{94}$$

We will remove the MDD subscript on $G(\cdot|\cdot)$ from now on when it is clear from the context. It is instructive to consider the limit $T \to \infty$. In this limit we should have that $G(h|T) = 1$ for all h. Using the eigenvalue equations (73), (92), we thus deduce the following interesting sums.

$$1 = \begin{cases} 2e^{-\frac{\mu h}{\sigma^2}} \sum_{n=1}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n} & \mu < \frac{\sigma^2}{h}, \\ \frac{2}{e} \sum_{n=1}^{\infty} \frac{\sin\theta_n}{\theta_n} + \frac{3}{e} & \mu = \frac{\sigma^2}{h}, \\ 2e^{-\frac{\mu h}{\sigma^2}} \left[\sum_{n=0}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n} - \frac{\sinh^3\eta}{\eta - \cosh\eta \sinh\eta} \right] & \mu > \frac{\sigma^2}{h}. \end{cases} \tag{95}$$

These sums have been verified numerically. We first consider the simplest case, that of $\mu = 0$. It turns out that this case can be handled exactly.

2.5.3 $\mu = 0$

In this case both the the discrete and continuous formulations give the same result. We see that when $\mu = 0$, the eigenvalue condition (73) is solved by $\theta_n = (n - \frac{1}{2})\pi$. Thus, we have that

$$\begin{aligned} G(h|T) &= 2\sum_{n=1}^{\infty} \frac{\sin(n-\frac{1}{2})\pi}{(n-\frac{1}{2})\pi} \left(1 - e^{-\frac{\sigma^2(n-1/2)^2\pi^2 T}{2h^2}}\right), & (96) \\ &= \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})} \left(1 - e^{-\frac{\sigma^2(n+1/2)^2\pi^2 T}{2h^2}}\right). & (97) \end{aligned}$$

It is tempting to compute dG/dh to get the density and then compute whatever expectations we wish to obtain. Unfortunately, this is tricky since we cannot take the derivative into the summation as the summation is not absolutely convergent. Infact, doing this will lead to an error. Instead, however, we can use Lemma 1.1 to obtain whatever expectations we wish. It turns out that getting the expected value of the MDD can be done in closed form.

$$\begin{aligned} E[MDD] &= \int_0^\infty dh\ G(h|T) = \frac{2}{\pi} \int_0^\infty dh \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})} \left(1 - e^{-\frac{\sigma^2(n+1/2)^2\pi^2 T}{2h^2}}\right), & (98) \\ &= 2\sigma\sqrt{T} \int_0^\infty dh \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})} \left(1 - e^{-\frac{(n+1/2)^2}{2h^2}}\right). & (99) \end{aligned}$$

Defining the constant γ by

$$\gamma = \int_0^\infty dh \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+\frac{1}{2})} \left(1 - e^{-\frac{(n+1/2)^2}{2h^2}}\right), \tag{100}$$

we see that

$$E[MDD] = 2\sigma\gamma\sqrt{T}. \tag{101}$$

In particular we have answered one part of the initial question of how does MDD depend on T. In otherwords, how should one scale MDD so that we can obtain a converging quantity. The exact computation of γ seems to be challenging. A numerical integration gave $\gamma \approx 0.6276$ and a simulation of the MDD gave $\gamma \approx 0.6226$, well within statistical fluctuations of the numerically computed value. To summarize the statistics we have so far, in the $\mu = 0$ case,

Statistic	Behavior
Standard Deviation	$\sigma\sqrt{T}$
Range	$2\sqrt{\frac{2}{\pi}}\sigma\sqrt{T}$
MDD	$2\gamma\sigma\sqrt{T}$

We see that all these statistics are basically proportional to each other in this limit. Note that $\sqrt{2/\pi} \approx 0.8$ so the range is considerably larger in expectation that the MDD. A more challenging computation would be $E\left[1/MDD\right]$. However, we will push on to the case of more general μ.

2.5.4 $\mu < 0$

After applying the eigenvalue conditions and taking the integral of $G(h|T)$ to get the expectation, we arrive at

$$E\left[MDD\right] = \int_0^\infty dh\ G(h|T) = 2\int_0^\infty dh\ e^{-\frac{\mu h}{\sigma^2}} \sum_{n=1}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}}\right). \tag{102}$$

Making a change of variables to $u = -\mu h/\sigma^2$, we find that

$$E\left[MDD\right] = -2\frac{\sigma^2}{\mu}\int_0^\infty du\ e^{u} \sum_{n=1}^{\infty} \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}}\right), \tag{103}$$

where $\tan\theta_n = -\theta_n/u$. It is clear that

$$E\left[MDD\right] = -\frac{2\sigma^2}{\mu} Q_n\left(\alpha^2\right), \tag{104}$$

for some function $Q_n(\cdot)$, where $\alpha = \mu\sqrt{T/2\sigma^2}$. The numerical computation of $Q_n(x)$ is not a straightforward task. The summation in the integrand is a function of u that decreases faster than e^{-u}. Since the magnitude of the n^{th} term in the summation is approximately $1/n$, we need to take $\Omega(e^u)$ terms in the summation to make sure that the next term left out has magnitude less than the size of the sum. Thus the efficient computation of this integral is computationally non trivial. The table in Appendix A gives approximate values of $Q_n(x)$ for various values of x, computed using an extensive numerical integration. Intermediate values could be obtained using interpolation or the asymptotic behavior discussed below.

We know that $Q_n(x) \to \gamma\sqrt{2x}$ when $x \to 0^+$, since in this limit, we must recover the $\mu \to 0$ behavior. We get the behavior in the $\alpha \to -\infty$ limit by noting that $R \geq MDD \geq -L$ where L is the low. Taking expectations, and using (46), we see that for all α,

$$\frac{\alpha^2}{2} + \frac{Q_R(-\alpha)}{2} \leq Q_n(\alpha^2) \leq Q_R(-\alpha). \tag{105}$$

Asymptotically, as $\alpha \to -\infty$, this yields

$$\alpha^2 + \frac{1}{4} \leq Q_n(\alpha^2) \leq \alpha^2 + \frac{1}{2}, \tag{106}$$

from which we deduce that $Q_n(x) \to x + \epsilon(x)$ where $\frac{1}{4} \leq \epsilon(x) \leq \frac{1}{2}$. Since $Q_n(x)$ is a monotonically increasing function of x (as the expected MDD has to increase when T increases), we

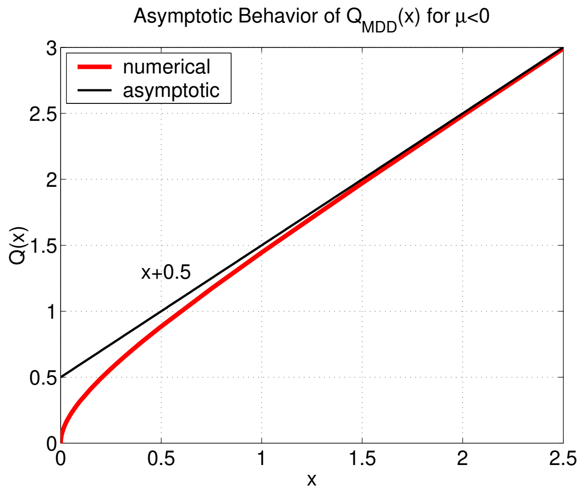

Figure 2: Asymptotic behavior of $Q_n(x)$.

conclude that $\epsilon(x) \to D_\infty$ for some constant D_∞, with $\frac{1}{4} \leq D_\infty \leq \frac{1}{2}$. We can write $E[MDD] \geq E[R|H \to L]P[H \to L]$, where $A \to B$ is used to denote that B occurs after A. Since $\mu < 0$, $P[H \to L] \to 1$ as $T \to \infty$. Conditioning on $H \to L$ will selectively pick higher ranges (when $L \to H$, the low cannot be very negative as $\mu < 0$) and so $E[R|H \to L] \geq E[R]$. Thus we see that asymptotically, $E[MDD] \geq E[R]$. This implies that asymptotically, $E[MDD] = E[R]$ and therefore that $D_\infty = \frac{1}{2}$. This fact is also verified by numerical computation of $Q_n(x)$. Thus, we have

$$Q_n(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ x + \frac{1}{2} & x \to \infty. \end{cases} \tag{107}$$

The asymptotic behavior is illustrated in Figure 2.

2.5.5 $\mu > 0$

In this case, for $h > \sigma^2/\mu$ in the integral, the third case for L adds another term. Thus we find that

$$E[MDD] = \int_0^\infty dh\, G(h|T), \tag{108}$$

$$= 2\int_0^\infty dh\, e^{-\frac{\mu h}{\sigma^2}} \sum_{n=1}^\infty \frac{\sin^3\theta_n}{\theta_n - \cos\theta_n \sin\theta_n}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}}\right) - 2\int_{\frac{\sigma^2}{\mu}}^\infty dh\, e^{-\frac{\mu h}{\sigma^2}} \frac{\sinh^3\eta}{\eta - \cosh\eta\sinh\eta}\left(1 - e^{-\frac{\mu^2 T}{2\sigma^2\cosh^2\eta}}\right). \tag{109}$$

The second integral can be reduced by a change of variables $u = \eta(h)$ as follows. Since $\tanh u = \sigma^2 u/\mu h$ we find that

$$\frac{dh}{du} = \frac{\sigma^2}{\mu}\frac{\cosh u \sinh u - u}{\sinh^2 u}, \tag{110}$$

hence, the second integral reduces to

$$-\frac{\sigma^2}{\mu}\int_0^\infty du\ e^{-\frac{u}{\tanh u}}\sinh u\left(1-e^{-\frac{\mu^2 T}{2\sigma^2\cosh^2 u}}\right). \tag{111}$$

Changing variable in the first integral to $u=\mu h/\sigma^2$, we arrive at

$$E\left[MDD\right]=2\frac{\sigma^2}{\mu}\int_0^\infty du\quad\left[e^{-u}\sum_{n=0}^{\infty}\frac{\sin^3\theta_n}{\theta_n-\cos\theta_n\sin\theta_n}\left(1-e^{-\frac{\mu^2 T}{2\sigma^2\cos^2\theta_n}}\right)+\right.$$
$$\left. e^{-\frac{u}{\tanh u}}\sinh u\left(1-e^{-\frac{\mu^2 T}{2\sigma^2\cosh^2 u}}\right)\right], \tag{112}$$

where $\tan\theta_n=\theta_n/u$. It is clear that

$$E\left[MDD\right]=\frac{2\sigma^2}{\mu}Q_p\left(\alpha^2\right), \tag{113}$$

for some function $Q_p(\cdot)$, where once again, $\alpha=\mu\sqrt{T/2\sigma^2}$. The bound (105) is still valid, but not very useful. The numerical computation of $Q_p(x)$ is relatively straightforward, as the e^{-u} term in the integrand makes it well behaved for the purposes of numerical integration. We know that $Q_p(x)\to\gamma\sqrt{2x}$ when $x\to 0^+$. We now consider the other asymptotic limit, namely $\alpha\to\infty$. We will evaluate the two contributions to $Q_p(x)$ separately. First consider $I_1(x)$ given by

$$I_1(x)=\int_0^\infty du\ e^{-u}\sum_{n=0}^{\infty}\frac{\sin^3\theta_n}{\theta_n-\cos\theta_n\sin\theta_n}\left(1-e^{-\frac{x}{\cos^2\theta_n}}\right). \tag{114}$$

Since $0\le\cos^2\theta_n\le 1$ and $x\to\infty$, the term in brackets is rapidly approaching 1. Since e^{-u} is rapidly decreasing, we interchange the summation with the integration and after changing variables in the integral to $v=\theta_n(u)$ and using the identity

$$du=\frac{\cos v\sin v-v}{\sin^2 v}dv, \tag{115}$$

we arrive at

$$I_1(x)=\sum_{n=0}^{\infty}\int_{n\pi}^{n+\frac{1}{2}\pi}dv\ e^{-\frac{v}{\tan v}}\sin v\left(1-e^{-\frac{x}{\cos^2\theta_n}}\right). \tag{116}$$

After translating each integral by $n\pi$ and bringing the summation back into the integral, the summation is a geometric progression which can be done in closed form to give

$$I_1(x)=\int_0^{\frac{\pi}{2}}dv\quad\frac{e^{-\frac{v}{\tan v}}\sin v\left(1-e^{-\frac{x}{\cos^2\theta_n}}\right)}{1+e^{-\frac{\pi}{\tan v}}}, \tag{117}$$

and so $(1-e^{-x})\beta_1\le I_1(x)\le\beta_1$ where β_1 is given by

$$\beta_1=\int_0^{\frac{\pi}{2}}dv\quad\frac{e^{-\frac{v}{\tan v}}\sin v}{1+e^{-\frac{\pi}{\tan v}}}, \tag{118}$$

and thus we see that $I_1(x)$ rapidly converges to the constant β_1. β_1 can be evaluated numerically to give $\beta_1 = 0.4575$. A numerical calculation of $I_1(x)$ for large x directly from (114) yielded 0.4578. Now consider the second term given by $I_2(x)$,

$$I_2(x) = \int_0^\infty du\ e^{-\frac{u}{\tanh u}} \sinh u \left(1 - e^{-\frac{x}{\cosh^2 u}}\right). \tag{119}$$

The term in brackets is the only place where x appears. When x is large, this term is very close to 1 until u gets large enough so that $\cosh u \sim x$, from which point the term in brackets rapidly decreases to 0. The term multiplying the term in brackets is always less than $\frac{1}{2}$ and rapidly increases from 0 to $\frac{1}{2}$. Thus we write

$$\begin{aligned} I_2(x) &= \int_0^\infty du\ \left(e^{-\frac{u}{\tanh u}} \sinh u - \frac{1}{2} + \frac{1}{2}\right)\left(1 - e^{-\frac{x}{\cosh^2 u}}\right), & (120)\\ &= \frac{1}{2}\int_0^\infty du\ \left(1 - e^{-\frac{x}{\cosh^2 u}}\right) - \int_0^\infty du\ \left(\frac{1}{2} - e^{-\frac{u}{\tanh u}} \sinh u\right) + & (121)\\ &\quad \int_0^\infty du\ e^{-\frac{x}{\cosh^2 u}} \left(\frac{1}{2} - e^{-\frac{u}{\tanh u}} \sinh u\right). & (122)\end{aligned}$$

It is not hard to show that the third integral approaches zero as $x \to \infty$ since the first term is small when u is small and the second term is small when u is large. The second integral is a constant β_2, independent of x and can be evaluated numerically to give $\beta_2 = 0.4575$, which is (numerically) equal to β_1. We suspect that $\beta_2 = \beta_1$ but the proof has been elusive. Nevertheless, β_2 is a constant which numerically appears equal to β_1.

We will get bounds for the first integral. Since $\cosh u \geq \frac{1}{2}e^u$ and for $u \geq A$, $\cosh u \leq \frac{1}{2}e^{\lambda(A)u}$ where $\lambda(A) = 1 + e^{-2A}/A$, denoting the first integral by $F(x)$, we immediately get the following bounds.

$$A\left(1 - e^{-\frac{x}{\cosh^2 A}}\right) + \int_A^\infty du\ \left(1 - e^{-4xe^{-2\lambda(A)u}}\right) \leq 2F(x) \leq \int_0^\infty du\ \left(1 - e^{-4xe^{-2u}}\right), \tag{123}$$

which hold for any A. A change of variables to $v = xe^{-2\lambda(A)u}$ in the lower bound and $v = xe^{-2u}$ in the upper bound then leads to the following bounds,

$$A\left(1 - e^{-\frac{x}{\cosh^2 A}}\right) + \frac{1}{2\lambda}\int_0^{xe^{-2\lambda A}} \frac{du}{u}\left(1 - e^{-4u}\right) \leq 2F(x) \leq \frac{1}{2}\int_0^x \frac{du}{u}\left(1 - e^{-4u}\right). \tag{124}$$

We can get an asymptotic form as follows. Suppose $x > 1$, then the following identity holds,

$$\int_0^x \frac{du}{u}\left(1 - e^{-4u}\right) = \int_0^1 \frac{du}{u}\left(1 - e^{-4u}\right) + \log x - \int_1^x \frac{du}{u} e^{-4u}. \tag{125}$$

When $x \to \infty$, the last converges term to $-Ei(-4)$ which can be computed numerically (see for example [Gradshteyn and Ryzhik, 1980]). The first term can be evaluated numerically and so as $x \to \infty$, we arrive at

$$\int_0^x \frac{du}{u}\left(1 - e^{-4u}\right) = \log x + C, \tag{126}$$

where numerically we compute $C \approx 1.9635$. Applying this identity to the upper bound and, for fixed A, to the lower bound, we eventually arrive at

$$\frac{1}{2\lambda(A)}(\log x + C) - Ae^{-\frac{x}{\cosh^2 A}} \leq 2F(x) \leq \frac{1}{2}\log x + \frac{C}{2}. \tag{127}$$

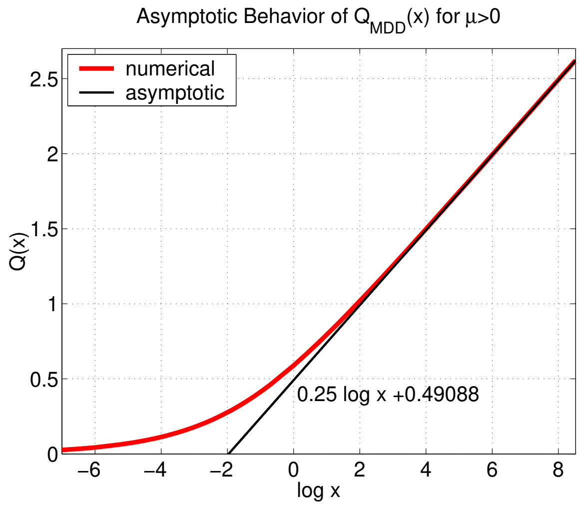

Figure 3: Asymptotic behavior of $Q_p(x)$.

Since A was arbitrary, it can be chosen to grow with x, for example $\frac{1}{2}(1+\epsilon)\log x$, in which case $\lambda(A) \to 1$ and the second term goes to 0, and so the upper and lower bounds approach each other. Thus we conclude that asymptotically as $x \to 0$,

$$F(x) \to \frac{1}{4}\log x + D, \tag{128}$$

where $D = \frac{C}{4} \approx 0.49088$. Putting all this together, noting that $Q_p(x) = I_1(x) + I_2(x)$, we find for the behavior of $Q_p(\cdot)$,

$$Q_p(x) \to \begin{cases} \gamma\sqrt{2x} & x \to 0^+, \\ \frac{1}{4}\log x + D & x \to \infty. \end{cases} \tag{129}$$

where $D \approx 0.49088$, and we have used the fact that $\beta_1 \approx \beta_2$. The asymptotic behavior is illustrated in Figure 3.

2.6 Correction for the Discrete Sampling

Practically, we can only sample the Brownian motion at a discrete set of points. Suppose that the sampling is at intervals Δt. The result will be a MDD that will have a negative bias, since the true MDD could have occured between two points that were not on the sampling grid. Rogers and Satchell consider exactly such a problem with respect to estimating the range of the Brownian motion from a discrete sampling.

The MDD will be given by $S - I$ where S is a local maximum of the sampled random walk at time t_1 and I is a local minimim at $t_2 > t_1$. Let $I_1 = \inf_{t\in(t_2-\Delta t, t_2+\Delta t)} X(t)$, and $S_1 = \sup_{t\in(t_1-\Delta t, t_1+\Delta t)} X(t)$, be the true local extrema near these points, and further suppose that $S_1 - I_1$ would still yield a maximal drawdown. Under these assumptions, the analysis of [Rogers and Satchell, 1991] applies, and writing $S_1 = S + \Delta$ and $I_1 = I + \tilde{\Delta}$, they derive asymptotic approximations to the expected values of Δ and $\tilde{\Delta}$ as $\Delta t \to 0$,

$$E[\Delta] = E\left[\tilde{\Delta}\right] \to \beta_{RS}\sigma\sqrt{\Delta t} \qquad \text{where} \qquad \beta_{RS} = \sqrt{2\pi}\left(\frac{1}{4} - \frac{(\sqrt{2}-1)}{6}\right). \tag{130}$$

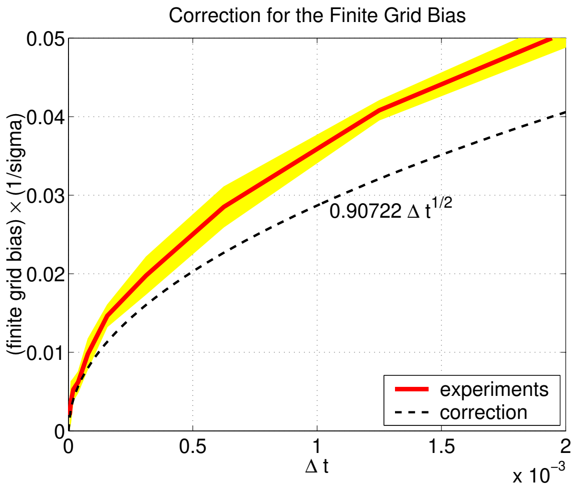

Figure 4: Comparison of the finite grid bias normalized by σ to the correction $2\beta_{RS}\sqrt{\Delta t}$.

Thus the sampled expected MDD will be negatively biased by an amount $2\beta_{RS}\sigma\sqrt{\Delta t}$. Notice that the lowest order term has no μ dependence. By simulation, we can compute $E\,[MDD]$ for various parameters μ, σ, T and Δt. Comparing to the theoretical calculation, we thus obtain the bias experimentally as a function $B(\mu, \sigma, T, \Delta t)$. Then the function $B(\mu, \sigma, T, \Delta t)/\sigma$ should approach $2\beta_{RS}\sqrt{\Delta t}$, independent of μ, σ, T. Figure 4 plots $B(\mu, \sigma, T, \Delta t)/\sigma$ averaged over σ chosen in a range from 0.3–10 (solid red curve). The spread is illustrated by the yellow shading, and the suggested correction term $2\beta_{RS}\sqrt{\Delta t}$ is also plotted (dashed black curve). As can be noted, the correction is quite accurate, especially as $\Delta t \to 0$.

A Table of Numerical Values for $Q(\cdot)$

x	$Q_p(x)$, $\mu > 0$	x	$Q_n(x)$, $\mu < 0$
$x \to 0$	$\gamma\sqrt{2x}$	$x \to 0$	$\gamma\sqrt{2x}$
0.0005	0.019690	0.0005	0.019965
0.0010	0.027694	0.0010	0.028394
0.0015	0.033789	0.0015	0.034874
0.0020	0.038896	0.0020	0.040369
0.0025	0.043372	0.0025	0.045256
0.0050	0.060721	0.0050	0.064633
0.0075	0.073808	0.0075	0.079746
0.0100	0.084693	0.0100	0.092708
0.0125	0.094171	0.0125	0.104259
0.0150	0.102651	0.0150	0.114814
0.0175	0.110375	0.0175	0.124608
0.0200	0.117503	0.0200	0.133772
0.0225	0.124142	0.0225	0.142429
0.0250	0.130374	0.0250	0.150739
0.0275	0.136259	0.0275	0.158565
0.0300	0.141842	0.0300	0.166229
0.0325	0.147162	0.0325	0.173756
0.0350	0.152249	0.0350	0.180793
0.0375	0.157127	0.0375	0.187739
0.0400	0.161817	0.0400	0.194489
0.0425	0.166337	0.0425	0.201094
0.0450	0.170702	0.0450	0.207572
0.0500	0.179015	0.0475	0.213877
0.0600	0.194248	0.0500	0.220056
0.0700	0.207999	0.0550	0.231797
0.0800	0.220581	0.0600	0.243374
0.0900	0.232212	0.0650	0.254585
0.1000	0.243050	0.0700	0.265472
0.2000	0.325071	0.0750	0.276070
0.3000	0.382016	0.0800	0.286406
0.4000	0.426452	0.0850	0.296507
0.5000	0.463159	0.0900	0.306393
1.5000	0.668992	0.0950	0.316066
2.5000	0.775976	0.1000	0.325586
3.5000	0.849298	0.1500	0.413136
4.5000	0.905305	0.2000	0.491599
10.0000	1.088998	0.2500	0.564333
20.0000	1.253794	0.3000	0.633007
30.0000	1.351794	0.3500	0.698849
40.0000	1.421860	0.4000	0.762455
50.0000	1.476457	0.5000	0.884593
150.0000	1.747485	1.0000	1.445520
250.0000	1.874323	1.5000	1.970740
350.0000	1.958037	2.0000	2.483960
450.0000	2.020630	2.5000	2.990940
1000.0000	2.219765	3.0000	3.492520
2000.0000	2.392826	3.5000	3.995190
3000.0000	2.494109	4.0000	4.492380
4000.0000	2.565985	4.5000	4.990430
5000.0000	2.621743	5.0000	5.498820
$x \to \infty$	$\frac{1}{4}\log x + 0.49088$	$x \to \infty$	$x + \frac{1}{2}$

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