FE2707 Expected value/mathematical expectation

The expected value plays a large role in ones ability to determine the reasonable outcome of a gamble, which in turn should determine how much one would be willing to pay for said gamble. Two examples will serve to illustrate. Suppose that one is to determine the price of a stock tomorrow by throwing a fair six sided die, the price being the value thrown. How much should one be willing to pay for this stock today? Suppose that an individual will live for t more years with probability $p(t)$ and wishes to buy a life insurance policy worth one million dollars. What monthly premium should the insurance company charge?

One approach to answering these questions is through the expected value of the gamble, which we illustrate with the stock example. The stock takes on values (in dollars) in the set $\{1, 2, 3, 4, 5, 6\}$, each with probability $\frac{1}{6}$. If the gamble is taken many times, or if one bets on many such stocks, say 600, then about 100 will be priced at 1, 100 at 2, etc. The total value will be $100 \times (1+2+3+4+5+6) = \$2100$, an "average" of about \$3.50 per stock. Thus, a risk neutral player should be willing to pay \$3.50 for this stock, less if risk averse, and more if risk seeking. As will be seen below, the mathematical expectation of this gamble is exactly \$3.50. This connection between the expectation and the intuitive notion of "average" is what motivates the formal definition, and in certain cases, one can show that the average and the expectation approach each other with arbitrary precision as the number of times one takes the gamble becomes large. In short, the expectation is a powerful tool that serves to represent the outcome of a (perhaps complicated) gamble by a single value.

Definition of Expectation. The formal version of a gamble is a random variable X that can take on values ω in some set, Ω. The two cases considered here are when Ω is discrete/countable (as with the stock example), or when Ω is some subset of the real line (as with the life insurance example). In the discrete case, a probability mass function $p(\omega_i)$ defines the probability that $X = \omega_i$. In the continuous case, a probability density function $p(\omega)d\omega$ defines the probability that $X \in [\omega, \omega + d\omega)$. Let $f(X)$ be any function of the random variable X, sometimes called the payoff function. Then, the mathematical expectation of f, $E[f(X)]$, is defined by

$$E[f(X)] = \sum_{\omega_i \in \Omega} p(\omega_i) f(\omega_i) \quad \text{(Discrete)} \qquad E[f(X)] = \int_{\omega_i \in \Omega} d\omega \; p(\omega) f(\omega) \quad \text{(Continuous)} \tag{1}$$

whenever the summation or integral satisfies certain convergence properties (see for example DeGroot and Schervish, 2002, ch.4), otherwise it does not exist. In words, multiply each possible payoff by the probability of obtaining that payoff and then take the sum. When $f(X)$ is a return on an investment, one usually refers to the expected return.

Examples.

(i) *Stock example.* $\Omega = \{1, 2, 3, 4, 5, 6\}$, with $p(\omega_i) = \frac{1}{6}$. $f(X) = X$ is the price of the stock. The expected stock price is given $E[f(X)] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$.

(ii) *Life insurance example.* $\Omega = [0, \infty)$. Let $p(t)$ be the probability density that $\omega = t$, i.e. that the insuree dies at time t. Suppose that the net profit (factoring in the present value of the premiums minus the pay-out) is given by $f(t)$. Then, the expected profit is given by $E[f(t)] = \int_0^\infty dt\ p(t)f(t)$, which will be a function of the premium. By increasing the premium, the insurance company makes more expected profit per policy sold, but the quantity of such policies sold will be lower. The insurance company will thus try to maximize total profit taking into account this trade off.

(iii) *St. Petersberg paradox.* The following gamble was invented by Daniel Bernoulli, 1700–1782. Person A tosses a coin until the first head appears. If the first head appears at toss n, A pays B 2^n dollars. How much should B be willing to pay to play this gamble? Let X, the random variable be the toss on which the first head appears. then $\Omega = \{1, 2, 3, \ldots\}$, with probabilities $\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. The payoff $f(X)$ is given by $\{2, 4, 8, \ldots\}$, and so the expected value of the payoff is $\sum f(\omega_i)p(\omega_i) = 1 + 1 + 1 + \ldots = \infty$. This is an example where the expectation does not exist, and so it is not clear how much this gamble is worth based on the expectation.

Some useful properties of the expectation.

(i) *Linearity.* If the payoff is the linear sum of two payoffs that may each depend on different random variables, then the expected payoff is the linear sum of the two expected payoffs.

(ii) *Product.* If the payoff is the product of two payoffs that each depend on different *independent* random variables, then the expected payoff is the product of the expected payoffs.

(iii) *Central Limit Theorem.* If the same gamble is played independently many times, then the average of the payoffs will approach the mathematical expectation, provided certain mathematical conditions are met.

While the expectation gives a reasonable characterization for the outcome of a gamble, it can be used to characterize its risk as well. Many measures of risk based on the expectation can be devised, the most often used being the variance. If the payoff function is $f(X)$ then the variance is defined by $Var(X) = E[f(X)^2] - E[f(X)]^2$. More details regarding the expectation can be found in DeGroot and Schervish (2002, ch.4) and Feller (1968, ch.9) and more technical discussions regarding probability and expectations can be found in Billingsley (1986, ch.4). The expectations of certain well know distributions can also be found in these references, as well as techniques for computing expectations. The definitions can be generalized to *conditional* expectations, and expectations of functions of more than one random variable, which can be found in the further reading.

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Further Reading

Billingsley, P. (1986). *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics. Wiley.

DeGroot, M. H. and Schervish, M. J. (2002). *Probability and Statistics.* Addison–Wesley, Reading, Massachusetts.

Devore, J. L. (1999). *Probability and Statistics for Engineering and the Sciences.* Duxbury Press.

Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, volume 1. John Wiley & Sons, New York.

Larsen, R. and Marx, M. (1986). *An Introduction to Mathematical Statistics.* Prentice-Hall, New Jersey.

Ross, S. M. (2001). *A First Course in Probability.* Prentice Hall, New Jersey.

See also: Averages (computations), Bayes theorem and risk analysis, Central limit theorem, Decision tree analysis, Martingales, Probability (fundamentals), Risk-return tradeoff.