Temporal Difference Learning

Reading

- Sutton, Richard S., and Barto, Andrew G. Reinforcement learning: An introduction. MIT press, 2018.
  - http://www.incompleteideas.net/book/the-book-2nd.html
  - Chapters 6.1-6.4, 7
- David Silver lecture on Dynamic Programming
  - https://www.youtube.com/watch?v=PnHCvfgC_ZA&t=585s
  - Second part of the lecture (on TD learning)

Overview

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- Dynamic programming (DP) requires full knowledge of the underlying MDP
  - Only possible for simple systems
- Monte Carlo methods require a lot of data to estimate state values
  - Also, not really online since need to wait for each episode to finish (and get the final reward)
    - Some tasks are continuous, others have very long episodes
  - Re-estimating state values for adapting policies requires yet more data
- Temporal Difference (TD) learning is the best of both worlds
  - Essentially a hybrid approach between the two
  - Main idea behind Q-learning also

Online State Value Estimation

- Suppose we have a fixed policy π and would like to estimate state values $v_\pi(s)$
- Suppose we start with some estimates $V(s)$ and would like to update them online
- With Monte Carlo methods, we need to wait until the end of each episode
  - Already saw the incremental (off-policy) implementation:

$$V^k(s) = V^{k-1}(s) + \frac{\rho_{t:T,k}}{\sum_{j=1}^{k} \rho_{t:T,j}} \left(G_{t,k} - V^{k-1}(s) \right)$$

- Can replace the ρ term with a "learning rate" α

$$V^k(s) = V^{k-1}(s) + \alpha \left(G_{t,k} - V^{k-1}(s) \right)$$

  - where α is now a hyperparameter

Online State Value Estimation

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- Suppose we don't want to wait until the end of the episode
  - i.e., instead of using G_t, we'd like to use each R_t
  - Can update policy after every step (much more efficient)
- In Monte Carlo learning, the value estimate is updated based on the difference between new return and current estimate

$$\alpha\left(G_{t,k} - V^{k-1}(s)\right)$$

- How do we adapt this idea to the case of a one-step reward?
  - What property does v_π have?
  - Bellman equation: $v_\pi(s) = \mathbb{E}_\pi[R_{t+1} + \gamma v_\pi(S_{t+1}) | S_t = s]$
  - Can replace the return with a bootstrapped estimate:

$$R_{t+1} + \gamma V^{k-1}(S_{t+1})$$

TD State Value Estimation

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- Update $V(S_t)$ after each step in an episode

$$V'(S_t) = V(S_t) + \alpha[R_{t+1} + \gamma V(S_{t+1}) - V(S_t)]$$

- Called a bootstrapping method because it is based in part on our existing estimates

Tabular TD(0) for estimating v_π

Input: the policy π to be evaluated
Algorithm parameter: step size $\alpha \in (0, 1]$
Initialize $V(s)$, for all $s \in \mathcal{S}^+$, arbitrarily except that $V(terminal) = 0$

Loop for each episode:
  Initialize S
  Loop for each step of episode:
    $A \leftarrow$ action given by π for S
    Take action A, observe R, S'
    $V(S) \leftarrow V(S) + \alpha[R + \gamma V(S') - V(S)]$
    $S \leftarrow S'$
  until S is terminal

TD as a hybrid method

- Similar to MC, TD updates estimates based on the difference between new and predicted data
  - A standard approach in general estimation theory
  - In some sense, this is a Bayesian method
    - Our current estimate of R_{t+1} is the prior
- TD is also similar to DP, since it uses the Bellman equation
  - Makes use of the Markov property and the MDP structure
  - Implicitly estimating the MDP structure
- We'll see that TD has hyper-parameters that can shift it along the "line" between DP and MC

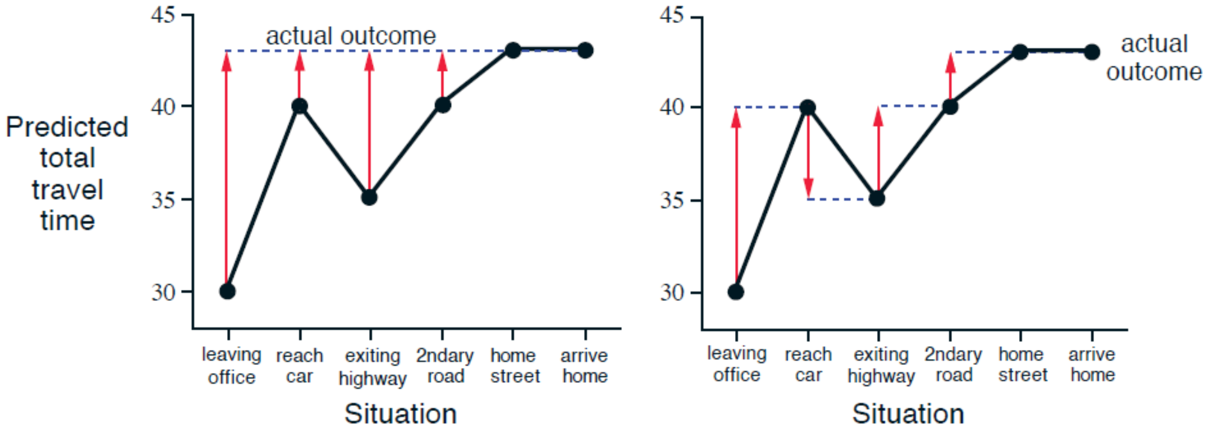
Example: Driving Home

- Suppose you leave your office at 6 on Friday
  - You have an ETA from previous trips
  - However, unforeseen events delay you
- You have pre-existing predicted time-to-go's for each situation (state)

State	*Elapsed Time (minutes)*	*Predicted Time to Go*	*Predicted Total Time*
leaving office, friday at 6	0	30	30
reach car, raining	5	35	40
exiting highway	20	15	35
2ndary road, behind truck	30	10	40
entering home street	40	3	43
arrive home	43	0	43

Example: Driving Home

- TD allows you to update your estimates after each event
  - E.g., you reach the car, and it is raining
    - You update your ETA based on the new information
- MC may introduce very large changes since it doesn't factor in intermediate states
  - Value updates may have great variance
    - An outlier data point, e.g., a slow truck, may cause a big change in the MC estimates

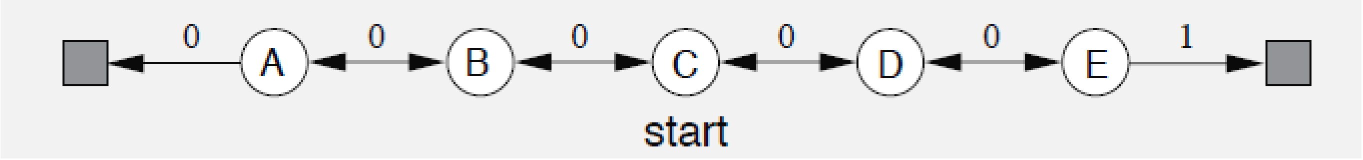


Advantages of TD Prediction

- No need to know the MDP, unlike DP
- Can be performed online, unlike Monte Carlo methods
- Usually more data-efficient than Monte Carlo methods since current estimate $V(S_t)$ can act as a prior
- Can update the state values after every step
  - May bring significant benefits if we also adapt the policy
  - A better policy may lead to seeing better rewards and faster learning overall
- MC minimizes the square error on the training set
  - TD converges to the maximum-likelihood estimate
- Both MC and TD converge to the true values given enough data
  - Proof is a bit technical, relies on stochastic processes

Advantages of TD Prediction, Example

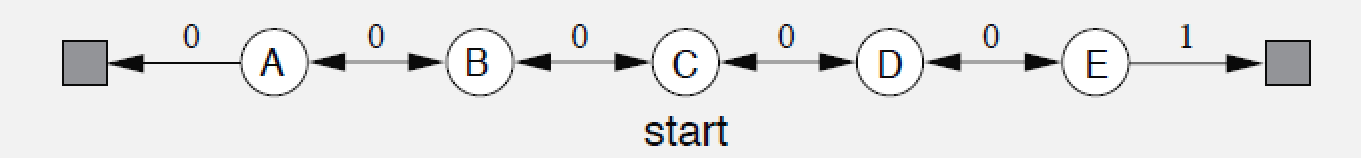
- Consider the following Markov Reward Process
  - There's a 0.5 chance of taking each transition
  - Reward of 1 only when we reach the right square



- What are the values of the various states?
  - Clearly, $v(C) = 0.5$ (equal chance of reaching each side)
  - Use matrix form of Bellman equation: $v(\boldsymbol{s}) = \boldsymbol{R} + \boldsymbol{P}v(\boldsymbol{s})$
  - Where $\boldsymbol{P} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{R} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$

Advantages of TD Prediction, Example

- Consider the following Markov Reward Process
  - 0.5 chance of taking each transition
  - Reward of 1 only when we reach the right square



- Need to use iterative policy evaluation method
  - Why?
  - Matrix $\boldsymbol{I} - \boldsymbol{P}$ is not invertible
    - Has an eigenvalue of 0
- Without discounting, state values are:

$$v_\pi(C) = 0.5, v_\pi(A) = \frac{1}{6}, v_\pi(B) = \frac{2}{6}, v_\pi(D) = \frac{4}{6}, v_\pi(E) = \frac{5}{6}$$

Advantages of TD Prediction, Example

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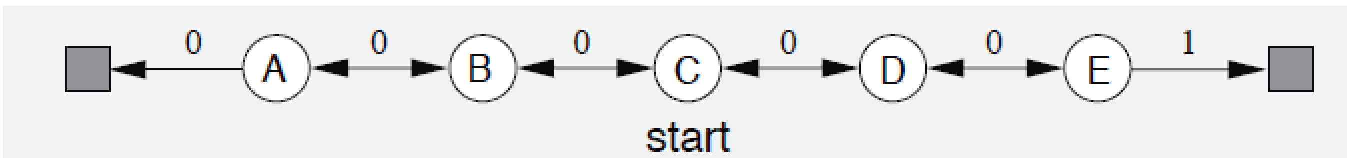
- Consider the following Markov Chain (with 0.5 chance of taking each transition)



- Without discounting, state values are:

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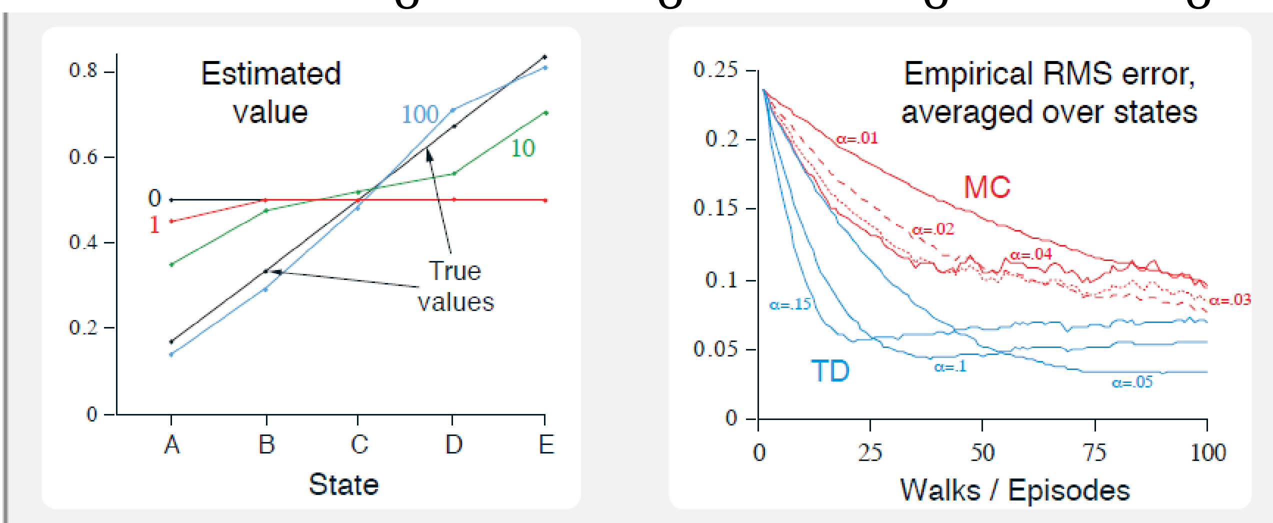
TD converges after ~100 episodes



TD Convergence Properties

- How does the learning rate α affect convergence?
  - Smaller α means slower convergence
  - Larger α means faster convergence but algorithm converges with bigger estimation error
- In order to truly converge to the optimal values, one needs to decrease α progressively
  - Will discuss in more detail when we get to Q learning

TD Batch Estimation

- Suppose we have fixed data from N episodes
  - Each episode is the usual $S_1, A_1, R_2, \ldots$
- Can apply TD estimation in batch fashion
  - Iterate through the episodes until convergence
  - TD guaranteed to converge
  - Guaranteed to converge to the true values as $N \rightarrow \infty$

Example: Difference between MC and TD

- Suppose we have an unknown MDP and observe the following 8 episodes with rewards
  - $A, 0, B, 0$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 0$
- What is your guess for the value at B?
  - A guess of 6/8 seems reasonable
- What is your guess for the value at A?
  - Both 6/8 and 0 seem reasonable

Example: Difference between MC and TD

- Suppose we have an unknown MDP and observe the following 8 episodes with rewards
  - $A, 0, B, 0$
  - $B, 1$
  - $B, 1$
  - $B, 1$

  $B, 1$

  $B, 1$

  $B, 1$

  $B, 0$
- Both MC and TD output 6/8 for $v(B)$
  – Why?
  – When we have a terminal state, TD is essentially MC
- If you use a constant α, you won't actually converge
  – Will bounce around 6/8
  – In true MC, $\alpha = 1/(1 + k)$

Example: Difference between MC and TD, cont'd

- Suppose we have an unknown MDP and observe the following 8 episodes with rewards
  - $A, 0, B, 0$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 0$
- MC will output 0 for $v(A)$
  - Why?
  - Only run that visited A had a return of 0

Example: Difference between MC and TD, cont'd

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- Suppose we have an unknown MDP and observe the following 8 episodes with rewards
  - $A, 0, B, 0$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 1$
  - $B, 0$
- TD will output 6/8 for $v(A)$
  - Why?
  - Eventually, $V(B)$ will converge to 6/8
  - When we process episode 1 after that (with $\alpha = 0.1$):

$$V'(A) = V(A) + \alpha\big(0 + V(B) - V(A)\big) = 3/40$$

    - This example assumes currently $V(A) = 0$ for simplicity
  - Keep iterating and $V(A)$ will bounce around $V(B)$

SARSA: On-policy TD Control

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- On-policy control idea is the same as before
  - Estimate the state/action values
  - For each state, choose the action with the highest value
- Action value recursion is the same as the state one:

$$Q'(S_t, A_t) = Q(S_t, A_t) + \alpha[R_{t+1} + \gamma Q(S_{t+1}, A_{t+1}) - Q(S_t, A_t)]$$

  - Requires also knowing the next action A_{t+1}
    - This makes it on-policy
  - Uses every quintuplet $S_t, A_t, R_{t+1}, S_{t+1}, A_{t+1}$
    - Hence the name
- To ensure exploration, still need ϵ-greedy policies

SARSA: On-policy TD Control, cont'd

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- After every step, alternate between policy evaluation and policy iteration
- Guaranteed to converge
  - (will see a sketch of the proof for Q-learning)

Sarsa (on-policy TD control) for estimating $Q \approx q_*$

Algorithm parameters: step size $\alpha \in (0, 1]$, small $\varepsilon > 0$
Initialize $Q(s, a)$, for all $s \in \mathcal{S}^+, a \in \mathcal{A}(s)$, arbitrarily except that $Q(terminal, \cdot) = 0$

Loop for each episode:
    Initialize S
    Choose A from S using policy derived from Q (e.g., ε-greedy)
    Loop for each step of episode:
        Take action A, observe R, S'
        Choose A' from S' using policy derived from Q (e.g., ε-greedy)
        $Q(S, A) \leftarrow Q(S, A) + \alpha\big[R + \gamma Q(S', A') - Q(S, A)\big]$
        $S \leftarrow S'$; $A \leftarrow A'$;
    until S is terminal

n-step bootstrapping

- MC methods wait for the return at the end of the episode
- TD updates its estimates and policies after every step
- TD learning may be noisy as it updates estimate too frequently
  - Smoothing over multiple steps would bring better results
- Can we have something in between?
  - Something that updates estimates/policies after n steps?
  - Yes, n-step bootstrapping!
  - Same idea as TD learning but applied over n steps

n-step return

- The MC return is

$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{T-t-1} R_T$$

- The TD return is just R_{t+1}
- What is the n-step TD return?

$$G_{t:t+n} = R_{t+1} + \gamma R_{t+2} + \cdots + \gamma^{n-1} R_{t+n}$$

- The MC evaluation recursion is

$$V'(S_t) = V(S_t) + \alpha\big(G_t - V(S_t)\big)$$

- The TD recursion is

$$V'(S_t) = V(S_t) + \alpha\big(R_{t+1} + \gamma V(S_{t+1}) - V(S_t)\big)$$

- What is the n-step recursion?

$$V'(S_t) = V(S_t) + \alpha\big(G_{t:t+n} + \gamma^n V(S_{t+n}) - V(S_t)\big)$$

n-step TD Prediction

- What is the n-step recursion?

$$V'(S_t) = V(S_t) + \alpha\big(G_{t:t+n} + \gamma^n V(S_{t+n}) - V(S_t)\big)$$

- Note that we need to wait for n steps to receive $G_{t:t+n}$
  - As $n \to \infty$, n-step TD converges to MC
- Larger n allow us to get a better estimate of $v_\pi(s)$
  - Adding stability at the expense of slower convergence

n-step TD for estimating $V \approx v_\pi$

Input: a policy π
Algorithm parameters: step size $\alpha \in (0, 1]$, a positive integer n
Initialize $V(s)$ arbitrarily, for all $s \in \mathcal{S}$
All store and access operations (for S_t and R_t) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$:
  | If $t < T$, then:
  | Take an action according to $\pi(\cdot|S_t)$
  | Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
  | If S_{t+1} is terminal, then $T \leftarrow t+1$
  | $\tau \leftarrow t - n + 1$ (τ is the time whose state's estimate is being updated)
  | If $\tau \geq 0$:
  | $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
  | If $\tau + n < T$, then: $G \leftarrow G + \gamma^n V(S_{\tau+n})$ ($G_{\tau:\tau+n}$)
  | $V(S_\tau) \leftarrow V(S_\tau) + \alpha\,[G - V(S_\tau)]$
  Until $\tau = T - 1$

Example: Random Walk

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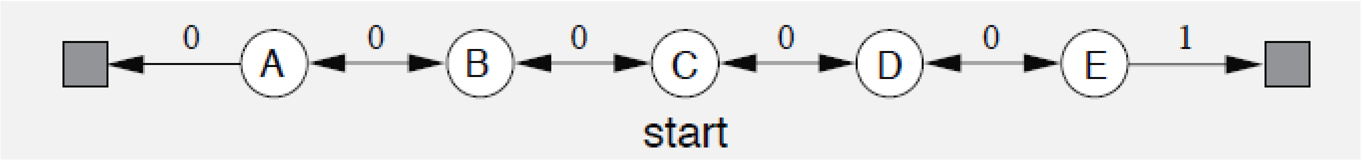
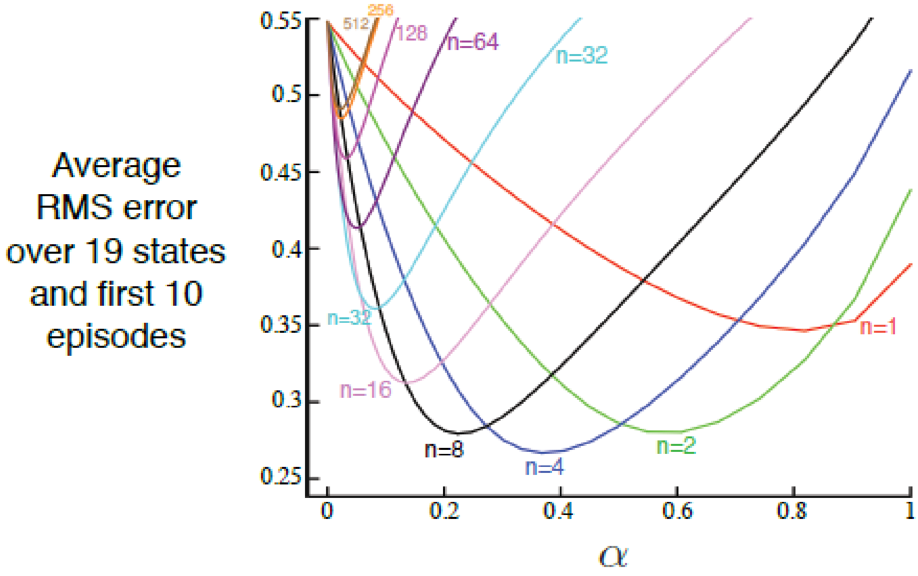
- Recall random walk example



  - Suppose we have 19 states instead of 5, i.e., 9 on each side
  - And suppose reward is -1 on the left

- Lowest error for $n = 4$
  - Best trade-off in this case
- MC ($n = \infty$) is pretty much the worst



n-step SARSA: On-policy Control

- Same idea as before
  - Estimate the state/action values
  - For each state, choose the action with the highest value
- Action value recursion is the same as the state one:

$$Q'(S_t, A_t) = Q(S_t, A_t) + \alpha[G_{t:t+n} + \gamma^n Q(S_{t+n}, A_{t+n}) - Q(S_t, A_t)]$$

n-step Sarsa for estimating $Q \approx q_*$ or q_π

Initialize $Q(s,a)$ arbitrarily, for all $s \in \mathcal{S}, a \in \mathcal{A}$
Initialize π to be ε-greedy with respect to Q, or to a fixed given policy
Algorithm parameters: step size $\alpha \in (0,1]$, small $\varepsilon > 0$, a positive integer n
All store and access operations (for S_t, A_t, and R_t) can take their index mod $n+1$

Loop for each episode:
  Initialize and store $S_0 \neq$ terminal
  Select and store an action $A_0 \sim \pi(\cdot|S_0)$
  $T \leftarrow \infty$
  Loop for $t = 0, 1, 2, \ldots$:
  | If $t < T$, then:
  | Take action A_t
  | Observe and store the next reward as R_{t+1} and the next state as S_{t+1}
  | If S_{t+1} is terminal, then:
  | $T \leftarrow t+1$
  | else:
  | Select and store an action $A_{t+1} \sim \pi(\cdot|S_{t+1})$
  | $\tau \leftarrow t - n + 1$ (τ is the time whose estimate is being updated)
  | If $\tau \geq 0$:
  | $G \leftarrow \sum_{i=\tau+1}^{\min(\tau+n,T)} \gamma^{i-\tau-1} R_i$
  | If $\tau + n < T$, then $G \leftarrow G + \gamma^n Q(S_{\tau+n}, A_{\tau+n})$ $(G_{\tau:\tau+n})$
  | $Q(S_\tau, A_\tau) \leftarrow Q(S_\tau, A_\tau) + \alpha\,[G - Q(S_\tau, A_\tau)]$
  | If π is being learned, then ensure that $\pi(\cdot|S_\tau)$ is ε-greedy wrt Q
  Until $\tau = T - 1$