# Quiz 2

60 Minutes

First Name: Solutions

Last Name: __________

RIN: __________

**NO COLLABORATION or electronic devices.**
**Any violations will result in an F.**
**No questions** allowed during the test unless you think there is a mistake.

## GOOD LUCK!

Circle at most one answer per question.
**10 points** for each correct answer.

You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

Final Score: ______ / 200

1. Suppose a goody-bag contains 3 candies. Candies come in three colors: red, green and blue. How many types of goody-bags are there in total?

   A $\binom{3}{2}$
   B 3!
   C $\binom{5}{2}$
   D 5!
   E None of the above.

   0 0 | 0 |
   Three 0's, 2 delimiters
   Five bits total

2. Suppose a goody-bag contains 3 candies. Candies come in three colors: red, green and blue. If all goody-bags are equally likely, how many goody-bags would I need to buy in order to guarantee I have a candy of each color?

   A $\binom{3}{2}$
   B 3!
   C $\binom{5}{2}$
   D 5!
   E None of the above.

   For any finite number, there's a non-zero probability to only have 2 colors.

3. Suppose FOCS has 10 students and every student tries to shake hands with as many other students as possible. How many handshakes need to occur in total to guarantee a repeat?

   A 10
   B 20
   C 35
   D 46
   E 91

   $\# \text{ handshakes} = \binom{10}{2} = \frac{10!}{2!\,8!} = \frac{10 \cdot 9}{2} = 45$

4. Let $X$ be a random variable and let the set of all outcomes be $\Omega$. What is $\sum_{x \in X(\Omega)} \mathbb{P}[X = x]$?

   A 0.5
   B 1
   C 1.5
   D 2
   E None of the above.

   Sum of all outcomes is 1.

5. Suppose $X(\Omega) = \{1, 2, 3\}$ and suppose $\mathbb{P}[X = 1 \vee X = 2] = 0.5$. What is $\mathbb{P}[X = 3]$?

   A 0.2
   B 0.3
   C 0.4
   D 0.5
   E 1

   $$\mathbb{P}[X=3] = 1 - \mathbb{P}[X \neq 3]$$
   $$= 1 - \mathbb{P}[X = 1 \vee X = 2]$$
   $$= 0.5$$

6. Suppose I toss three coins independently. What do we know?

   A At least two coins must match.
   B The probability that all coins match is 1/8.
   C The probability of at least one H is 1.
   D The probability of at least one T is 1.
   E None of the above.

Pigeonhole principle.
3 pigeons (tosses)
and 2 holes (coin values)

7. Suppose $X_1$ and $X_2$ are independent and uniform on $\{1, 2, 3, 4, 5\}$. What is $\mathbb{P}[X_1 + X_2 \leq 3]$?

   A $\frac{1}{25}$
   B $\frac{2}{25}$
   C $\frac{3}{25}$
   D $\frac{4}{25}$
   E $\frac{5}{25}$

$$A = \{(X_1 = 1, X_2 = 1), (X_1 = 1, X_2 = 2), (X_1 = 2, X_2 = 1)\}$$

$$\mathbb{P}[X_1 + X_2 \leq 3] = \frac{3}{25}.$$

SHOW WORK

8. Suppose $X_1$ and $X_2$ are independent and uniform on $\{1, 2, 3, 4, 5\}$. What is $\mathbb{E}[X_1 + X_2]$?

   A 3
   B 4
   C 5
   D 6
   E 7

$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{4}{5} + \frac{5}{5} = 3$$

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 6$$

SHOW WORK

9. Suppose $X_1$ is uniform on $\{1, 2, 3, 4, 5\}$. If $X_1 \geq 4$, then $X_2$ is uniform on $\{4, 5\}$; otherwise $X_2 = 5$. What is $\mathbb{E}[X_1 + X_2]$?

   A $\frac{30}{5}$
   B $\frac{33}{5}$
   C $\frac{36}{5}$
   D $\frac{39}{5}$
   E $\frac{42}{5}$

$\mathbb{E}[X_1] = 3$ (Question 8)

$$\mathbb{E}[X_2] = \mathbb{E}[X_2 | X_1 \geq 4] \cdot \mathbb{P}[X_1 \geq 4] + \mathbb{E}[X_2 | X_1 < 4] \cdot \mathbb{P}[X_1 < 4]$$

$$= \frac{2}{5} \cdot 4.5 + \frac{3}{5} \cdot 5 = \frac{24}{5}$$

$$\mathbb{E}[X_1 + X_2] = 3 + \frac{24}{5} = \frac{39}{5}$$

SHOW WORK

10. Suppose Submitty had a bug and randomly shuffled Quiz 2 grades. Assuming there are 200 students and all grades are different, what is the expected number of students who get their correct grade in Submitty?

   A 1 B 10 C 20 D 50 E None of the above.

$X_i = 1$ if student $i$ got their correct grade

$X = X_1 + \cdots + X_{200}$ (all students w/ correct grade)

$\mathbb{E}[X_i] = \frac{1}{200}$ (grades randomly distributed). So $\mathbb{E}[X] = \frac{200}{200} = 1$

11. Suppose the correct answer is not E on any of the 20 questions and you guess randomly among A-D. How many of the 20 questions do you expect to get right?

- [A] 1
- [B] 2
- [C] 3
- [D] 4
- [E] 5

$\mathbb{P}[\text{success}] = \frac{1}{4}$.

~~$\mathbb{P}[20$ successes~~

$E[20 \text{ trials}] = \frac{20}{4} = 5$

12. Suppose you answer A on all 20 questions. How many questions do you expect to get right?

- [A] 2
- [B] 3
- [C] 4
- [D] 5
- [E] It cannot be determined from the given information.

Depends on the correct answer distribution.
(correct answer could be always B)

13. Suppose it is sunny 1/10 of days in Troy. How much do you expect to wait until a sunny day?

- [A] 5 days
- [B] 10 days
- [C] 15 days
- [D] 20 days
- [E] None of the above.

$\mathbb{P}[\text{success}] = \frac{1}{10} = p$

$E[\text{waiting time}] = \frac{1}{p} = 10$

14. Suppose it is sunny 1/10 of days in Troy. Suppose it is always sunny in Philadelphia, except for the days when it is sunny in Troy. How many days is a Philadelphian expected to wait until a sunny day?

- [A] 10/9
- [B] 9/10
- [C] 2
- [D] 3
- [E] None of the above.

$\mathbb{P}[\text{success}] = \frac{9}{10}$

$E[\text{waiting time}] = \frac{10}{9}$

SHOW WORK

15. You are in Troy now. If the weather is not sunny, you travel to Philadelphia tomorrow; if it's not sunny in Philadelphia tomorrow, you go back to Troy the day after (and will go back and forth on non-sunny days). How many days do you expect to wait until a sunny day (assuming same probabilities of sunny days as in Question 14)?

[A] 91/91 [B] 100/91 [C] 190/91 [D] 290/91 [E] 182/91

$P_T = \frac{1}{10} + (1 + P_P) \cdot \frac{9}{10}$

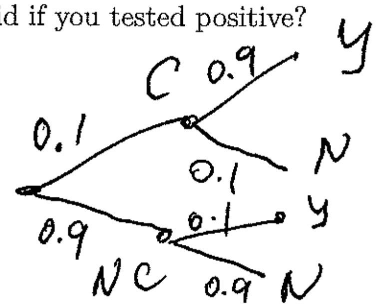
$P_P = \frac{9}{10} + (1 + P_T) \cdot \frac{1}{10} = 1 + \frac{1}{10} P_T$

$\Rightarrow P_T = 1 + \frac{9}{10} + \frac{9}{100} P_T \rightarrow P_T = \frac{190}{91}$

SHOW WORK

16. Suppose a covid test is correct 90% of the time and 10% of all people have covid. What is the probability that you have covid if you tested positive?

   - A 1/10
   - B 9/10
   - C 1/3
   - D 1/2
   - E None of the above.

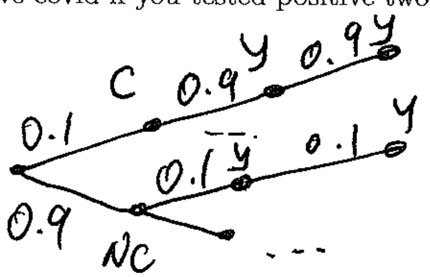



$$\mathbb{P}[C|Y] = \frac{\mathbb{P}[C \wedge Y]}{\mathbb{P}[Y]} = \frac{0.9 \cdot 0.1}{2 \cdot 0.9 \cdot 0.1} = \frac{1}{2}$$

SHOW WORK

17. Suppose a covid test is correct 90% of the time and 10% of all people have covid. What is the probability that you have covid if you tested positive two times independently?

   - A 50/100
   - B 50/90
   - C 81/100
   - D 81/90
   - E None of the above.




$$\mathbb{P}[C|2Y] = \frac{\mathbb{P}[C \wedge 2Y]}{\mathbb{P}[2Y]} = \frac{\frac{81}{1000}}{\frac{81+9}{1000}} = \frac{81}{90}$$

18. Suppose the correct answer is uniform on $\{A, B, C, D, E\}$. What is the probability that at least 2 of the 20 questions have the same letter for the correct answer?

   - A 0
   - B $\binom{20}{2}$
   - C 0.5
   - D 1
   - E None of the above.

Pigeonhole principle. 20 pigeons and 5 pigeonholes

19. If each question had 26 choices, and the correct answer is uniform on $\{A, \ldots, Z\}$, what is the probability that at least 2 of the 20 questions have the same letter for the correct answer?

   - A $\left(\frac{25}{26}\right)^{19} \times \left(\frac{24}{25}\right)^{18} \times \cdots \times \left(\frac{6}{7}\right)$
   - ~~B~~ $1 - \left(\frac{25}{26}\right)^{19} \times \left(\frac{24}{25}\right)^{18} \times \cdots \times \left(\frac{6}{7}\right)$
   - C $\left(\frac{25}{26}\right)^{19}$
   - D $1 - \left(\frac{25}{26}\right)^{19}$
   - E None of the above.

Birthday paradox problem

Correct probability is $1 - \left(\frac{25}{26}\right)^{19} \cdot \left(\frac{24}{25}\right)^{18} \cdots \left(\frac{7}{8}\right)$

20. $X \sim B(p_1)$ and $Y \sim B(p_2)$ are independent Bernoulli random variables. What is $\mathbb{E}[XY]$?

   - A 1/4
   - B $p_1 p_2$
   - C $p_1/p_2$
   - D $p_1 + p_2$
   - E None of the above.

X and Y are independent, so

$$\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y] = p_1 p_2$$

# Scratch