Quiz 1

60 Minutes

First Name: SOLUTIONS

Last Name: __________

RIN: __________

Section: __________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

Circle at most one answer per question.
10 points for each correct answer.

You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

Final Score: ______ / 200

1. What is a simpler expression for the set $A = \{n \mid n = 12k + 3m, k \in \mathbb{N}, m \in \mathbb{N}\}$?

   - $\boxed{A}$ $A = \{n \mid n = 3k, k \in \mathbb{N}\}$
   - $\boxed{B}$ $A = \{n \mid n = 12 + 3k, k \in \mathbb{N}\}$
   - $\boxed{C}$ $A = \{n \mid n = 12 + 3k, k \in \mathbb{Z}\}$
   - $\boxed{D}$ $A = \{n \mid n = 15 + 3k, k \in \mathbb{N}\}$
   - $\boxed{E}$ None of the above.

   $A = \{n \mid 3(4k + m)\ k, m \in \mathbb{N}\}$

   15, 18, 21, ...

2. How did we use the principle of well-ordering to prove that $\sqrt{2}$ is irrational?

   - $\boxed{A}$ We used it to assume $\sqrt{2}$ is rational.
   - $\boxed{B}$ We used it to conclude $\sqrt{2}$ has a factor of 3.
   - $\boxed{C}$ We used it to list the set of all pairs (p, q) such that $\sqrt{2} = p/q$.
   - $\boxed{D}$ We did not use the principle of well-ordering.
   - $\boxed{E}$ None of the above.

   We used it to select the smallest q^* st $\sqrt{2} = \frac{p^*}{q^*}$

3. What is the value of the expression $(p \wedge \neg p) \rightarrow (\neg p \wedge q)$?

   - $\boxed{A}$ Depends on the value of p.
   - $\boxed{B}$ Depends on the value of q.
   - $\boxed{C}$ Always true.
   - $\boxed{D}$ Always false.
   - $\boxed{E}$ None of the above.

   F → anything

4. IF you are in FOCS, THEN you must have taken CS1 AND you must have taken DS. Suppose you have not taken CS1. What do we know?

   - $\boxed{A}$ You are not in FOCS.
   - $\boxed{B}$ You are in DS.
   - $\boxed{C}$ The value of "you are in FOCS" depends on the value of "you have taken DS".
   - $\boxed{D}$ The value of "you have taken DS" depends on the value of "you are in FOCS".
   - $\boxed{E}$ None of the above.

   $p \rightarrow q \wedge r$

   $F \leftarrow F$

5. How many rows are there in the truth table of $(p \wedge q) \vee (\neg p \wedge q) \vee q$?

   - $\boxed{A}$ 2
   - $\boxed{B}$ 3
   - $\boxed{C}$ 4
   - $\boxed{D}$ 8
   - $\boxed{E}$ 16

   Two variables ⇒ 4 possible values

6. Suppose I want to prove $p \to q$. Which of the following proof techniques will work?

$\boxed{\text{A}}$ Assume p and show that it leads to a contradiction. ✓ False $\to$ anything

$\boxed{\text{B}}$ Assume q is false and show that p must be false. ✓ $\neg q \to \neg p$

$\boxed{\text{C}}$ Assume q is false and show it leads to a contradiction. ✓ anything $\to$ False

$\boxed{\text{D}}$ Use derivations to show that if q is false, then p must be false. ✓ $\neg q \to \neg p$

$\boxed{\text{E}}$ All of the above.

7. What is the negation of the claim: $\forall x \in \mathbb{Z} : \exists y \in \mathbb{N} : x > y$?

$\boxed{\text{A}}$ $\forall x \in \mathbb{Z} : \exists y \in \mathbb{N} : x \leq y$

$\boxed{\text{B}}$ $\forall x \in \mathbb{Z} : \forall y \in \mathbb{N} : x > y$

$\boxed{\text{C}}$ $\forall x \in \mathbb{N} : \exists y \in \mathbb{Z} : x > y$

$\boxed{\text{D}}$ $\exists x \in \mathbb{Z} : \forall y \in \mathbb{N} : x \leq y$

$\boxed{\text{E}}$ None of the above.

Flip quantifiers, and negate predicate

8. Suppose $p, q \in \mathcal{P}$ are prime ($\mathcal{P}$ is the set of all prime numbers). Which of the following is true?

$\boxed{\text{A}}$ $\forall p, q \in \mathcal{P} : pq - 1$ is prime.

$\boxed{\text{B}}$ $\forall p, q \in \mathcal{P} : pq - 1$ is not prime.

$\boxed{\text{C}}$ $\exists p, q \in \mathcal{P} : pq - 1$ is prime. $2 \times 3 - 1 = 5$ is prime

$\boxed{\text{D}}$ $\forall p, q \in \mathcal{P} : p + q$ is prime.

$\boxed{\text{E}}$ None of the above.

9. Suppose I try to prove $n^2 \leq 2^n, \forall n \geq 1$ using induction. What goes wrong?

$\boxed{\text{A}}$ The base case is false.

$\boxed{\text{B}}$ I need more base cases.

$\boxed{\text{C}}$ I cannot prove $P(n) \to P(n+1), \forall n \geq 1$. $3^2 > 2^3$.

$\boxed{\text{D}}$ I cannot prove $P(n+1) \to P(n), \forall n \geq 1$. Can't prove $P(2) \to P(3)$

$\boxed{\text{E}}$ Nothing goes wrong because the claim is true.

10. How would you *disprove* the claim: $\forall m \in \mathbb{N} : \exists n \in \mathbb{N} : m^2 = n$?

$\boxed{\text{A}}$ Show that $m^2 \neq n$ for all natural numbers m and n. Find a counterexample

$\boxed{\text{B}}$ Show that $m^2 \neq n$ for all integers m and n. $\exists m \in \mathbb{N} : \forall n \in \mathbb{N} : m^2 \neq n$

$\boxed{\text{C}}$ Find some $m, n \in \mathbb{N}$ for which $m^2 \neq n$.

$\boxed{\text{D}}$ Find some $m \in \mathbb{N}$ for which $m^2 \neq n$ for all $n \in \mathbb{N}$.

$\boxed{\text{E}}$ None of the above.

11. Consider the recurrence $T_0 = 1, T_n = T_{n-2} + 2$. What is T_{179}?

- $\boxed{A}$ 178
- $\boxed{B}$ 179
- $\boxed{C}$ 180
- $\boxed{D}$ It is not defined.
- $\boxed{E}$ None of the above.

$T_0 = 1,\quad T_2 = 3,\quad T_4 = 5, \ldots$
Only even elements are defined

12. Consider the set $\mathcal{S}$ defined as follows: (1) Base case: $1 \in \mathcal{S}$; (2) Constructor: $x \in \mathcal{S} \to x + 2 \in \mathcal{S}$. Which of the following cannot be the set $\mathcal{S}$?

- $\boxed{A}$ All odd natural numbers. ✓
- $\boxed{B}$ All odd integers. ✓
- $\boxed{C}$ All natural numbers. ✓
- $\boxed{D}$ $\mathbb{N} \cap \{n \mid n = 2k - 3, k \in \mathbb{N}\}$. ✓ $(-1 \in S \Rightarrow 1 \in S)$
- $\boxed{E}$ $\mathbb{N} \cap \{n \mid n = 2k, k \in \mathbb{N}\}$. ✗ 1 is not in this set

Show work

13. Define the predicate $P(n) : 3n^2 \leq n^3$. For which n is $P(n)$ true?

- $\boxed{A}$ $n \geq 1$ ✗ $3 > 1$
- $\boxed{B}$ $n \geq 2$ ✗ $12 > 8$
- $\boxed{C}$ $n \geq 3$ ✓ $27 \leq 27$
- $\boxed{D}$ Only for $n = 4$ and $n = 5$ ✗
- $\boxed{E}$ None of the above

Assume true for $n \geq 3$
$3(n+1)^2 = 3n^2 + 6n + 3 \leq n^3 + 3n^2 + 3n + 1 = (n+1)^3$

14. Which of the following proof techniques can be used to prove $n \leq 3^{n/3}, \forall n \geq 0$? Suppose $P(n) : n \leq 3^{n/3}$.

- $\boxed{A}$ Show $P(0)$ and $P(1)$ are true and show that $P(1) \wedge \cdots \wedge P(n) \to P(n+1), \forall n \geq 0$.
- $\boxed{B}$ Show that $P(n) \to P(n+1), \forall n \geq 0$. ✗ No base case
- $\boxed{C}$ Show that $P(0) \wedge P(1) \wedge \cdots \wedge P(n) \to P(n+1), \forall n \geq 0$. ✗ No base case
- $\boxed{D}$ Define $Q(n) = P(0) \wedge P(1) \wedge \cdots \wedge P(n)$ and show that $Q(n) \to Q(n+1), \forall n \geq 0$. ✗ No base case
- $\boxed{E}$ None of the above.

15. What can we say about this claim: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R} : xy = y$?

- $\boxed{A}$ True $x = 1$
- $\boxed{B}$ False
- $\boxed{C}$ Depends on the value of x
- $\boxed{D}$ Depends on the value of y
- $\boxed{E}$ None of the above.

SHOW WORK

16. What is another expression for the set $A \cap \overline{(A \cap B)}$?

[A] $A \cap \overline{B}$
[B] $A \cup \overline{B}$
[C] $A \cap \overline{A} \cap B$
[D] $A \cup \overline{A} \cup A \cap B$
[E] None of the above.

$A \cap \overline{(A \cap B)} = A \cap (\overline{A} \cup \overline{B})$
$= \underbrace{(A \cap \overline{A})}_{\phi} \cup A \cap \overline{B} = A \cap \overline{B}$

17. What is a formal way to say "Every positive real distance is realized by some points on the plane"?

[A] $\exists x \in \mathbb{R} : \forall x_1, x_2, y_1, y_2 \in \mathbb{R} : x = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
[B] $\exists x \in \mathbb{R} : \exists x_1, x_2, y_1, y_2 \in \mathbb{R} : x = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
[C] $\forall x \in \mathbb{R} \cap \{z \mid z > 0, z \in \mathbb{R}\} : \forall x_1, x_2, y_1, y_2 \in \mathbb{R} : x = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
[D] $\forall x \in \mathbb{R} \cap \{z \mid z > 0, z \in \mathbb{R}\} : \exists x_1, x_2, y_1, y_2 \in \mathbb{R} : x = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
[E] None of the above.

distance

positive reals pairs of points in $\mathbb{R}^2$

SHOW WORK

18. Let $A_n = 2A_{n-1} - 1$ and $A_0 = 2$. What is a general formula for A_n for $n \geq 1$?

[A] $A_n = 2n - 1$
[B] $A_n = 2n + 1$
[C] $A_n = 2^n$
[D] $A_n = 2^n + 1$
[E] None of the above.

$A_0 = 2^0 + 1 = 2.$ ✓
Assume true for A_n.
$A_{n+1} = 2A_n - 1 = 2 \cdot 2^n + 2 - 1 = 2^{n+1} + 1$

SHOW WORK

19. Suppose I create a new type of rooted tree, called perfect binary tree (PBT), as follows:

① The tree with one vertex is a PBT. **[base case]**

② If PBTs T_1 and T_2 with roots r_1 and r_2 have the same structure, then linking r_1 and r_2 to a new root r gives a new PBT with root r. **[constructor]**

③ No other tree is a PBT. **[minimality]**

What do we know about PBTs (recall RBT stands for a rooted binary tree)?

[A] The number of vertices of any PBT is some number n such that $n = 2^k - 1, k \in \mathbb{N}$.
[B] The sets of all PBTs and all RBTs are the same.
[C] All RBTs are PBTs.
[D] All PBTs have an even number of vertices.
[E] None of the above.

Base case: $k=1$. $n=1$.
Tree w/ 1 vertex. ✓
Induction step:
T_1 and T_2 have the same # of nodes, say n_1, so
$n = 2n_1 + 1 = 2 \cdot 2^k - 2 + 1 = 2^{k+1} - 1$

SHOW WORK

20. Suppose a rooted binary tree has 8 vertices in its left subtree (ignoring the root). How many vertices are in the right subtree (ignoring the root) if the tree has 17 links in total?

[A] 9 [B] 10 [C] 11 [D] 12 [E] It cannot be determined.

Total vertices is 1 + links, so 18
root + left + right = 18 $\Rightarrow$ right = 9
$\begin{matrix} \| & \| \\ 1 & 8 \end{matrix}$

Scratch