Sums and Asymptotics

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 9

Overview

- Maximum Substring Sum
- Computing Sums
- Asymptotics: Big-Theta, Big-Oh and Big-Omega
- Integration Method

Maximum Substring Sum

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- Look at this sequence of numbers:

$$1, -1, -1, 2, 3, 4, -1, -1, 2, 3, -4, 1, 2, -1, -2,1$$

- What is the largest sum of 7 consecutive numbers?

$$2 + 3 + 4 - 1 - 1 + 2 + 3 = 12$$

  – This is known as the max substring sum

- More generally, compute the maximum substring sum for

$$a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n$$

  – where n measures the size/length of the input

Maximum Substring Sum, cont'd

- Can you come up with an algorithm for max substring sum?
  1. Iterate over all pairs (i, j) of start and end positions.
     - Brute-force but effective and easy to analyze
     - How many loops do we have?
     - 3 loops: one loop for each of i and j, and 1 loop to calculate sum
  2. Iterate over all starting positions i and ending positions $j > i$.
     - More efficient than 1
     - How many loops do we have?
     - 2 loops: one loop each over all i and all j
  3. Divide and conquer
     - Divide array into two halves and recursively calculate max in each half
     - Also look at max sum that contains the middle
  4. Suppose you are keeping track of the current cumulative sum
     - What happens if a sum is negative (assuming positive numbers exist)?
     - Should reset sum to next number
     - If current sum is larger than the largest so far, set largest to current

Maximum Substring Sum, cont'd

- Different algorithms have different runtimes (check book exercises for specific algorithms)
  - three-loop version: $T_1 = 2 + \sum_{i=1}^{n} \left[2 + \sum_{j=1}^{n} \left(5 + \sum_{k=i}^{j} 2\right)\right]$
    - What does $\sum_{i=1}^{n}$ mean?
    - Sum all entries, increasing i by 1 each time
  - two-loop version: $T_2(n) = 2 + \sum_{i=1}^{n} \left(3 + \sum_{j=i}^{n} 6\right)$
  - A recursive algorithm:

$$T_3(n) = \begin{cases} 3 & n = 1 \\ 2T_3\left(\frac{1}{2}n\right) + 6n + 9 & n > 1\ (even) \\ T_3\left(\frac{1}{2}(n+1)\right) + T_3\left(\frac{1}{2}(n-1)\right) + 6n + 9 & n > 1\ (odd) \end{cases}$$

  - A fast algorithm: $T_4(n) = 5 + \sum_{i=1}^{n} 10$
- But which one is fastest?

Evaluate Runtimes

- Let's plug in some values of n and see what happens

n	1	2	3	4	5	6	7	8	9	10
$T_1(n)$	11	29	58	100	157	231	324	438	575	737
$T_2(n)$	11	26	47	74	107	146	191	242	299	362
$T_3(n)$	3	27	57	87	123	159	195	231	273	315
$T_4(n)$	15	25	35	45	55	65	75	85	95	105

- Which algorithm is best?
  - Clearly, T_1 is worse than T_2 but hard to compare T_2 and T_3
  - T_4 seems best on most inputs
- We need:
  - Simple formulas for $T_1(n), \ldots, T_4(n)$: we need to compute sums and solve recurrences.
  - A way to compare runtime-*functions* that captures the essence of the algorithm.

Computing Sums: Tool 1: Constant Rule

- $S_1 = \sum_{i=1}^{10} 3$
  $$= 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 3 \times 10$$
- $S_2 = \sum_{i=1}^{10} j$
  $$= j + j + j + j + j + j + j + j + j + j = j \times 10$$
- $S_3 = \sum_{i=1}^{10} i$
  $$= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
  $$= \frac{1}{2} \times 10 \times 11$$
- The *index of summation* is i in these examples.
- **Constants (independent of summation index) can be taken outside the sum.**

$$S_1 = \sum_{i=1}^{10} 3 = 3 \sum_{i=1}^{10} 1 = 3 \times 10$$

$$S_2 = \sum_{i=1}^{10} j = j \sum_{i=1}^{10} 1 = j \times 10$$

Computing Sums: Tool 2: Addition Rule

$$S = \sum_{i=1}^{5}(i + i^2)$$

$$= (1 + 1^2) + (2 + 2^2) + (3 + 3^2) + (4 + 4^2) + (5 + 5^2) \quad \textbf{[rearrange terms]}$$

$$= (1 + 2 + 3 + 4 + 5) + (1^2 + 2^2 + 3^2 + 4^2 + 5^2)$$

$$= \sum_{i=1}^{5} i + \sum_{i=1}^{5} i^2$$

- **The sum of terms added together is the addition of the individual sums**

$$\sum_{i}(a(i) + b(i) + \cdots) = \sum_{i} a(i) + \sum_{i} b(i) + \cdots$$

Computing Sums: Tool 3: Common Sums

$$\sum_{i=k}^{n} 1 = n - k + 1$$

$$\sum_{i=1}^{n} f(x) = nf(x)$$

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \quad [r \neq 1]$$

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

$$\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$$

$$\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$$

Computing Sums: Tool 3: Common Sums, cont'd

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$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

$$\sum_{i=0}^{n} \frac{1}{2^i} = 2 - \frac{1}{2^n}$$

$$\sum_{i=1}^{n} \log i = \log n!$$

Computing Sums: Example

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$$\sum_{i=1}^{n}\left(1+2i+2^{i+2}\right)=$$

$= \sum_{i=1}^{n} 1 + \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 2^{i+2}$ **[addition rule]**

$= \sum_{i=1}^{n} 1 + 2\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} 2^{i}$ **[constant rule]**

$= n + 2 \times \frac{1}{2} n(n+1) + 4 \times \left(2^{n+1} - 1 - 1\right)$ **[common sums]**

$= n + n(n+1) + 2^{n+3} - 8$ **[algebra]**

Computing Sums: Tool 3: Nested Sum Rule

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- To compute a nested sum, start with the innermost sum and proceed outward

$$S_1 = \sum_{i=1}^{3} \sum_{j=1}^{3} 1$$
$$= \sum_{j=1}^{3} 1 + \sum_{j=1}^{3} 1 + \sum_{j=1}^{3} 1 = 3 + 3 + 3 = 9$$

  – Note that the j variables are local to each sum (same as in a loop in your code)

$$S_2 = \sum_{i=1}^{3} \sum_{j=1}^{i} 1$$
$$= \sum_{j=1}^{1} 1 + \sum_{j=1}^{2} 1 + \sum_{j=1}^{3} 1 = 1 + 2 + 3 = 6$$

- More generally
  – Using the fact that $\sum_{j=1}^{i} 1 = i$:

$$S(n) = \sum_{i=1}^{n} \sum_{j=1}^{i} 1 = \sum_{i=1}^{n} i = \frac{1}{2} n(n+1)$$

Computing a formula for T_2

$$T_2(n) = 2 + \sum_{i=1}^{n}\left(3 + \sum_{j=i}^{n} 6\right)$$ **[sum rule]**

$$= 2 + 3\sum_{i=1}^{n} 1 + \sum_{i=1}^{n}\sum_{j=i}^{n} 6$$ **[constant rule]**

$$= 2 + 3n + \sum_{i=1}^{n}\sum_{j=i}^{n} 6$$ **[common sum]**

$$= 2 + 3n + 6\sum_{i=1}^{n}\sum_{j=i}^{n} 1$$ **[constant rule]**

$$= 2 + 3n + 6\sum_{i=1}^{n}(n - i + 1)$$ **[innermost sum]**

$$= 2 + 3n + 6(n + (n-1) + \cdots + 1)$$ **[common sum]**

$$= 2 + 3n + 6 \times \frac{1}{2} n(n+1)$$ **[common sum]**

$$= 2 + 6n + 3n^2$$ **[algebra]**

Practice: Compute a Formula for the Sum: $\sum_{i=1}^{n}\sum_{j=1}^{i} ij$

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$$\sum_{i=1}^{n}\sum_{j=1}^{i} ij = \sum_{i=1}^{n}\sum_{j=1}^{i} ij \quad \text{[innermost sum]}$$

$$= \sum_{i=1}^{n} i\sum_{j=1}^{i} j \quad \text{[constant rule]}$$

$$= \sum_{i=1}^{n} i\,\frac{1}{2}i(i+1) \quad \text{[common sum]}$$

$$= \frac{1}{2}\sum_{i=1}^{n}\left(i^3 + i^2\right) \quad \text{[algebra, constant rule]}$$

$$= \frac{1}{2}\sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} i^2 \quad \text{[sum rule]}$$

$$= \frac{1}{8}n^2(n+1)^2 + \frac{1}{12}n(n+1)(2n+1) \quad \text{[common sums]}$$

$$= \frac{1}{12}n + \frac{3}{8}n^2 + \frac{5}{12}n^3 + \frac{1}{8}n^4 \quad \text{[algebra]}$$

Summary of Max Substring Sum Algorithms

- Runtimes

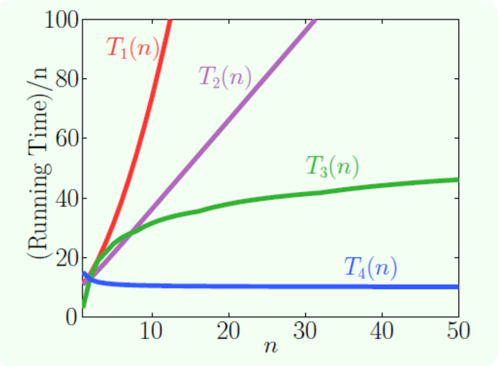
  $\boldsymbol{T_1(n)} = 2 + \frac{31}{6}n + \frac{7}{2}n^2 + \frac{1}{3}n^3$

  $\boldsymbol{T_2(n)} = 2 + 6n + 3n^2$

  $3n(\log_2 n + 1) - 9 \leq \boldsymbol{T_3(n)} \leq 12n(\log_2 n + 3) - 9$

  $\boldsymbol{T_4(n)} = 5 + 10n$

  – ("simple" formulas for $T_1(n), \ldots, T_4(n)$)



- **So, which algorithm is best?**
  – Computers solve problems with big inputs. We care about large n.
  – Compare runtimes *asymptotically* in the input size n. That is $n \to \infty$
  – Ignore additive and multiplicative constants (minutia). We care about *growth rate*.

- Algorithm 4 is *linear* in n, $\frac{T_4(n)}{n} \to$ constant.

Asymptotically Linear Functions: $\boldsymbol{\Theta(n)}$, big-Theta-of-n

- We say an algorithm runs in "big-Theta-of-n" time, i.e.,

$T \in \Theta(\boldsymbol{n})$, if there are positive constants c, C for which

$$c \cdot \boldsymbol{n} \leq T(\boldsymbol{n}) \leq C \cdot \boldsymbol{n}$$

$$\frac{T(n)}{n} \xrightarrow[n\to\infty]{} \begin{cases} \infty & T \in \omega(n), \quad \text{"}T > n\text{"} \\ constant > 0 & T \in \Theta(n), \quad \text{"}T = n\text{"} \\ 0 & T \in o(n), \quad \text{"}T < n\text{"} \end{cases}$$

- Linear means in $\Theta(n)$:

$$2n + 7, \quad 2n + 15\sqrt{n}, \quad 10^9 n + 3, \quad 3n + \log n, \quad 2^{\log_2 n + 4}$$

- Not linear means not in $\Theta(n)$:

$$10^{-9} n^2, \quad 10^9\sqrt{n} + 15, \quad n^{1.0001}, \quad n^{0.9999}, \quad n \log n, \quad \frac{n}{\log n}, \quad 2^n$$

- Other runtimes frequently appearing in practice

log	linear	loglinear	quadratic	cubic	super-polynomial	exponential	factorial	BAD
$\Theta(\log n)$	$\Theta(n)$	$\Theta(n \log n)$	$\Theta(n^2)$	$\Theta(n^3)$	$\Theta(n^{\log n})$	$\Theta(2^n)$	$\Theta(n!)$	$\Theta(n^n)$

General Asymptotics: $\Theta(f)$, big-Theta-of-f

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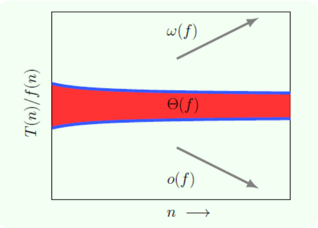
- Sometimes, we want to measure performance w.r.t. a specific function f

$$\frac{T(f)}{f(n)} \xrightarrow[n\to\infty]{} \begin{cases} \infty & T \in \omega(f), \quad "T > f" \\ constant > 0 & T \in \Theta(f), \quad "T = f" \\ 0 & T \in o(f), \quad "T < f" \end{cases}$$



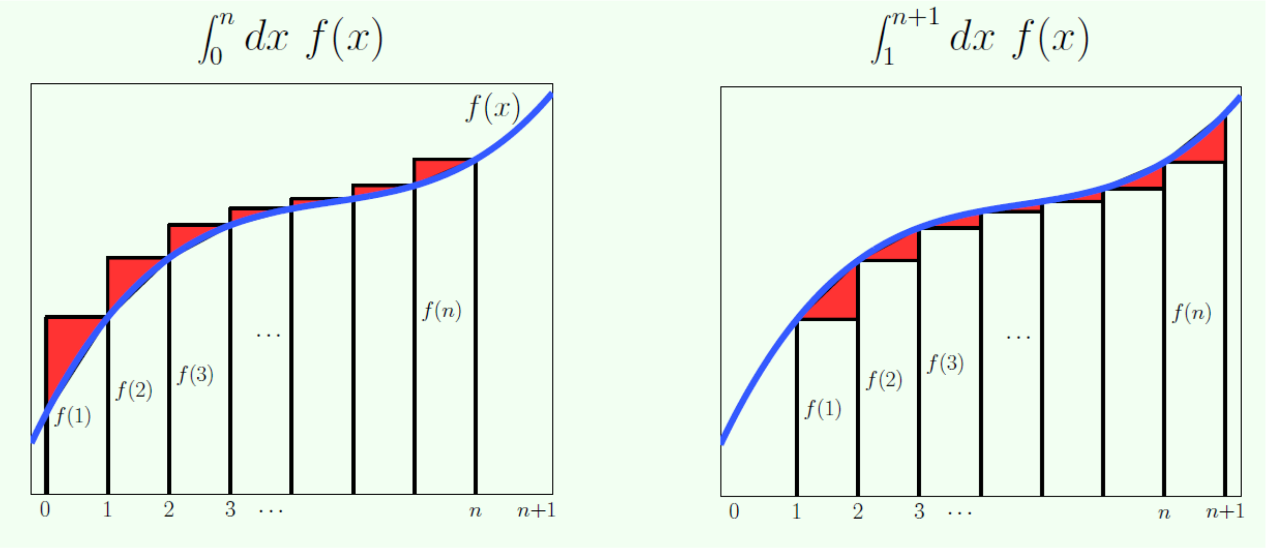
$T \in o(f)$	$T \in O(f)$	$T \in \Theta(f)$	$T \in \Omega(f)$	$T \in \omega(f)$
"$T < f$"	"$T \leq f$"	"$T = f$"	"$T \geq f$"	"$T > f$"
	$T(n) \leq Cf(n)$	$cf(n) \leq T(n) \leq Cf(n)$	$cf(n) \leq T(n)$	

- Examples:
  - For polynomials, growth rate is the highest order

$$\Theta(2n^2) = n^2$$
$$\Theta(n^2 + n\sqrt{n}) = \Theta(n^2)$$
$$\Theta(n^2 + \log^{256} n) = \Theta(n^2)$$
$$\Theta(n^2 + n^{1.99}\log^{256} n) = \Theta(n^2)$$

The Integration Method

- One application of big-Theta reasoning
  - You can approximate an integral with the upper and lower integration method



- *Theorem [Integration Bound]*. For a monotonically increasing function f,

$$\int_{m-1}^{n} f(x)dx \le \sum_{i=m}^{n} f(i) \le \int_{m}^{n+1} f(x)dx$$

  - (If f is monotonically decreasing, the inequalities are reversed.)

Integration for Quickly Getting Asymptotic Behavior

- **Integer Powers**. Set $f(x) = x^k$:

$$\sum_{i=1}^{n} i^k \approx \int_0^n x^k dx$$

$$\int_0^n x^k dx = \frac{n^{k+1}}{k+1}$$

$$\frac{n^{k+1}}{k+1} \in \Theta(n^{k+1})$$

Integration for Quickly Getting Asymptotic Behavior, cont'd

- **Stirling's Approximation for $\ln n$!**. Set $f(x) = \ln x$:

$$\ln n! = \sum_{i=1}^{n} \ln i \leq \int_{1}^{n+1} \ln x \, dx$$

$$\int_{1}^{n+1} \ln x \, dx =$$

$$= [x \ln(x) - x]_{1}^{n+1}$$
$$= (n+1)\ln(n+1) - n$$

- So finally:

$$\big((n+1)\ln(n+1) - n\big) \in \Theta(n \ln n)$$

Integration for Quickly Getting Asymptotic Behavior, cont'd

- **Analyzing a recurrence.** $T_1 = 1; T_n = T_{n-1} + n\sqrt{n} - \ln n$
  - First, unfold the recurrence:

$$\begin{aligned} T_n &= T_{n-1} + n\sqrt{n} - \ln n \\ T_{n-1} &= T_{n-2} + (n-1)\sqrt{n-1} - \ln(n-1) \\ &\vdots \\ T_3 &= T_2 + 3\sqrt{3} - \ln 3 \\ T_2 &= T_1 + 2\sqrt{2} - \ln 2 \end{aligned}$$

  - Sum all terms together (note that all $T_{n-1}, \ldots, T_1$ terms cancel out)

$$\begin{aligned} T_n &= 1 + 2\sqrt{2} + \cdots + n\sqrt{n} - (\ln 2 + \ln 3 + \cdots + \ln n) \\ &= \sum_{i=1}^{n} i\sqrt{i} - \sum_{i=1}^{n} \ln i \end{aligned}$$

    - Why does the 2nd sum start counting from 1?
  - We know that $\sum_{i=1}^{n} i\sqrt{i} \in \Theta\left(n^{5/2}\right)$
  - and $\sum_{i=1}^{n} \ln i = \ln n! \in \Theta(n \ln n)$
- What does that mean for $T(n) = \sum_{i=1}^{n} i\sqrt{i} - \sum_{i=1}^{n} \ln i$?

$$T(n) \in \Theta\left(n^{5/2}\right)$$