Proofs with Recursion

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 8

Overview

- Two Types of Questions About Recursive Sets
- Matched Parentheses
- Structural Induction
  - $\mathbb{N}$
  - Palindromes
  - Arithmetic Expressions
- Rooted Binary Trees (RBT)

Two Types of Questions About a Recursive Set

- Recursive definition of a set $\mathcal{A}$:

$$0 \in \mathcal{A}$$
$$x \in \mathcal{A} \rightarrow x + 4 \in \mathcal{A}$$

- What is $\mathcal{A}$?

$$\mathcal{A} = \{0,4,8,12,16, \dots\}$$

  - (i) What is in $\mathcal{A}$? Is some feature common to every element of $\mathcal{A}$? Is everything in $\mathcal{A}$ even?

$$x \in \mathcal{A} \rightarrow x \text{ is even (T)}$$

  - (ii) Is everything with some property in $\mathcal{A}$? Is every even number in $\mathcal{A}$?

$$x \text{ is even} \rightarrow x \in \mathcal{A} \text{ (F)}$$

- **Very, very** different statements!
  - Every leopard has 4 legs
  - Is everything with 4 legs a leopard?

Structural Induction

- Structural induction shows every member of a recursive set has a property, question (i)
- Consider the evolution of orcs
  - (Because computer scientists have nothing better to do)
  - The first two orcs had blue eyes
  - When two orcs mate, if they both have blue eyes, then the child has blue eyes
  - Do all orcs have blue eyes?
  - When could a green-eyed orc have arisen?
- **Structural Induction**
  - The ancestors have a trait
  - The trait is passed on from parents to children
  - Conclusion: Everyone today has that trait

Matched Parentheses $\mathcal{M}$

- Recursive definition of $\mathcal{M}$

$$\varepsilon \in \mathcal{M} \qquad\qquad [\textbf{basis}]$$

$$x, y \in \mathcal{M} \to [x] \bullet y \in \mathcal{M} \qquad [\textbf{constructor}]$$

- The strings in $\mathcal{M}$ are the matched (in the arithmetic sense) parentheses. For example:

[] (set $x = \varepsilon, y = \varepsilon$ to get $[\varepsilon]\varepsilon = []$)

[[]] (set $x = [], y = \epsilon$)

[][] (set $x = \varepsilon, y = []$)

- Let's list the strings in $\mathcal{M}$ as they are created

$$\mathcal{M} = \{\varepsilon, [], [[]], [][], [[]][], \ldots, s_n, \ldots\}$$

- To get s_n, we apply the constructor to two prior (not necessarily distinct) strings.

Strings in $\mathcal{M}$ Are Balanced

- Balanced means the number of opening and closing parentheses are equal
- The constructor,

$$x, y \in \mathcal{M} \to [x] \bullet y \in \mathcal{M}$$

  - adds one opening and closing parenthesis
- If the "parent" strings x and y are balanced, then the child $[x] \bullet y$ is balanced.
- (Orcs inherit blue eyes. Here, parents pass along balance to the children.)
- Just as all orcs will have blue eyes, all strings in $\mathcal{M}$ will be balanced.

Proof: Strings in $\mathcal{M}$ are Balanced

$$\begin{aligned} \mathcal{M} &= \{\varepsilon, [], [[]], [][], [[]][], \dots, s_n, \dots\} \\ &= \{s_1, s_2, s_3, \dots, s_n, \dots\} \end{aligned}$$

- What is the $P(n)$?
  - $P(n)$: string s_n is balanced, i.e., the number of '[' equals the number of ']'

Proof: Strings in $\mathcal{M}$ are Balanced, cont'd

$$\mathcal{M} = \{s_1, s_2, s_3, \dots, s_n, \dots\}$$

- $P(n)$: string s_n is balanced, i.e., the number of '[' equals the number of ']'
  - But is that the only property?
  - Every '[' is eventually followed by a corresponding ']'
- *Proof*. [Strong induction on n]

1. [**Base case**] $P(1)$ claims that $s_1 = \varepsilon$ is balanced. True.
2. [**Induction step**] Show that $P(1) \wedge \cdots \wedge P(n) \rightarrow P(n+1)$. Direct proof.
   - Assume $P(1) \wedge \cdots \wedge P(n)$: $s_1, \dots, s_n$ are all balanced
   - Show $P(n+1)$: s_{n+1} is balanced
   - s_{n+1} is the child of two *earlier* strings: $s_{n+1} = [s_k] \bullet s_l$ (**constructor rule**)
   - s_k, s_l appeared earlier than s_{n+1}, so they are balanced (induction hypothesis)
   - Therefore, s_{n+1} is balanced (you add parentheses around a balanced string)
3. By induction, $P(n)$ is T $\forall n \geq 1$

Proof: Strings in $\mathcal{M}$ are Balanced, cont'd

- **Question**. Is every balanced string in $\mathcal{M}$?
  - What about][?
- **Exercise**. Prove that [[ ] $\notin \mathcal{M}$

Structural Induction Overview

- Strong induction with recursively defined sets is called *structural induction*
- Let $\mathcal{S}$ be a recursive set. This means you have:
  - Base cases $s_1, \ldots, s_k$ that are in $\mathcal{S}$
  - Constructor rules that use elements in $\mathcal{S}$ to create a new element of $\mathcal{S}$
- Let $P(s)$ be a property defined for any element $s \in \mathcal{S}$. To show $P(s)$ for every element in $\mathcal{S}$, you must show:
  1. [**Base cases**] $P(s_1), P(s_2), \ldots, P(s_k)$ are T
  2. [**Induction step**] For *every* constructor rule, show:
     - IF P is T for the parents, THEN P is T for children
  3. By structural induction, conclude that $P(s)$ is T for all $s \in \mathcal{S}$
- **MUST** show for *every* base case
- **MUST** show for *every* constructor rule
- Structural induction can be used with *any* recursive set

Every Opening Parenthesis in $\mathcal{M}$ is Matched

- Example string in $\mathcal{M}$: [[ ]] [ ]
  - Three opening and three closing parentheses
- Going from left to right:

  [, opening = 1, closing = 0

  [, opening = 2, closing = 0

  ], opening = 2, closing = 1

  ], opening = 2, closing = 2

  [, opening = 3, closing = 2

  ], opening = 3, closing = 3
- Opening is always at least closing: parentheses are arithmetically matched
  - **Important Exercise.** Prove this by structural induction
  - Key step is to show that constructor preserves "matchedness"
- **Question.** Is every string of matched parentheses in $\mathcal{M}$?
- **Hard Exercise.** Prove this. (see Exercise 8.3)

Structural Induction on $\mathbb{N}$

- $\mathbb{N} = \{1,2,3, \dots\}$ is a recursively defined set:

$$1 \in \mathbb{N}$$
$$x \in \mathbb{N} \rightarrow x + 1 \in \mathbb{N}$$

- Consider any property of the natural numbers, for example

$$P(n)\text{: } 5^n - 1 \text{ is divisible by 4}$$

- Structural induction to prove $P(n)$ holds for every $n \in \mathbb{N}$:
  1. [**Prove for all base cases**] Only one base case $P(1)$.
  2. [**Prove every constructor rule *preserves*** $P(n)$] Only one constructor:
     - IF P is T for x (the parent), then P is T for $x + 1$ (the child).
  3. By structural induction, $P(n)$ is T $\forall n \in \mathbb{N}$
- That's just ordinary induction!

Palindromes $\mathcal{P}$

- Here's a nerdy palindrome: "Was it a rat I saw"
  - The same sequence of letters forwards and backwards
- Binary sequences

$$(01100)^R = 00110 \qquad \text{(not a palindrome)}$$
$$(0110)^R = 0110 \qquad \text{(a palindrome)}$$

- Recursive definition of palindromes $\mathcal{P}$
  - There are three base cases: $\varepsilon \in \mathcal{P}, 0 \in \mathcal{P}, 1 \in \mathcal{P}$
  - There are two constructor rules: $(i) x \in \mathcal{P} \rightarrow 0 \bullet x \bullet 0 \in \mathcal{P}$;
    $(ii) x \in \mathcal{P} \rightarrow 1 \bullet x \bullet 1 \in \mathcal{P}$
- Constructor *rules* preserves palindromicity:

$$(0 \bullet 0110 \bullet 0)^R = 001100$$
$$(1 \bullet 0110 \bullet 1)^R = 101101$$

- Therefore, we can prove by structural induction that all strings in $\mathcal{P}$ are palindromes
- **Hard Exercise.** Prove that all palindromes are in $\mathcal{P}$ (Exercise 8.7).

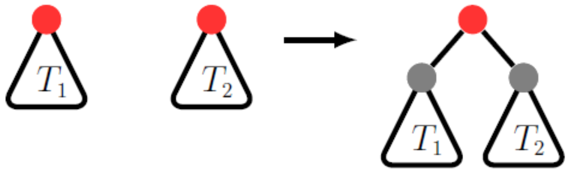
Arithmetic Expressions

- Fact known to all first-graders: ((1+1+1) × (1+1+1+1+1))=15
  - i.e., `value`((1+1+1)×(1+1+1+1+1))=15
- A recursive set of well formed arithmetic expression strings $\mathcal{A}_{ODD}$:
  - One base case: $1 \in \mathcal{A}_{ODD}$
  - There are two constructor rules: $(i)\ x \in \mathcal{A}_{ODD} \rightarrow (x + 1 + 1) \in \mathcal{A}_{ODD}$;
    $(ii)\ x, y \in \mathcal{A}_{ODD} \rightarrow (x \times y) \in \mathcal{A}_{ODD}$
  - For example,

$$1 \rightarrow (1+1+1) \rightarrow ((1+1+1)+1+1)$$
$$(1 \times 1) \qquad ((1 \times 1)+1+1)$$

- The constructors add 2 to the parent or multiply the parents.
- If the parents have odd value, then so does the child.
- Constructors preserve "oddness" → all strings in $\mathcal{A}_{ODD}$ have odd value

Rooted Binary Trees with $n \geq 1$ Vertices Have $n - 1$ Edges

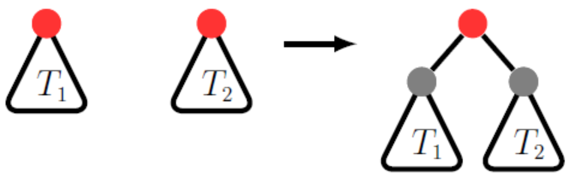


- The empty tree ε is an RBT.
- Disjoint RBTs T_1, T_2 give a new RBT by linking their roots to a new root.
- $P(T)$: if T is a rooted binary tree with $n \geq 1$ vertices, then T has $n - 1$ links.
  1. [**Base case**] $P(\varepsilon)$ is vacuously T because ε is not a tree with $n \geq 1$ vertices.
  2. [**Induction step**] Consider the constructors with parent RBTs T_1 and T_2
     - **Parents**: T_1 with n_1 vertices and l_1 edges and T_2 with n_2 vertices and l_2 edges.
     - **Child**: T with n vertices and l edges.
     - <u>Case 1</u>: $T_1 = T_2 = \varepsilon$.
       – Child is a single node with $n = 1, l = 0 = n - 1$
     - <u>Case 2</u>: $T_1 = \varepsilon; T_2 \neq \varepsilon$.
       – The child has one more node, $n = n_2 + 1$, and one more link:

$$l = l_2 + 1$$
$$= n_2 - 1 + 1 = n_2 = n - 1 \text{ [\textbf{induction hypothesis}]}$$

Rooted Binary Trees with $n \geq 1$ Vertices Have $n - 1$ Edges, cont'd

Rensselaer



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     - **Parents**: T_1 with n_1 vertices and l_1 edges and T_2 with n_2 vertices and l_2 edges.
     - **Child**: T with n vertices and l edges.
     - <u>Case 3</u>: $T_1 \neq \varepsilon; T_2 = \varepsilon$. (Similar to Case 2.)

       $n = n_1 + 1 \ AND$

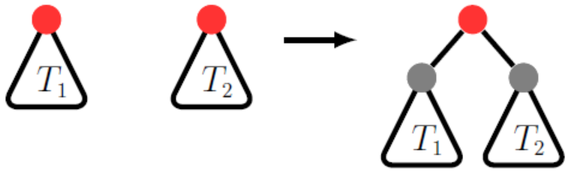
       $l = l_1 + 1$

       $= n_1 - 1 + 1 = n_1 = n - 1$ [**induction hypothesis**]
     - <u>Case 4</u>: $T_1 \neq \varepsilon; T_2 \neq \varepsilon$.
       - Now, $n = n_1 + n_2 + 1$ and there are two new links, so

       $l = l_1 + l_2 + 2$

       $= n_1 - 1 + n_2 - 1 + 2 = n_1 + n_2 = n - 1$ [**induction hypothesis**]

Rooted Binary Trees with $n \geq 1$ Vertices Have $n-1$ Edges, cont'd



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     - **Parents**: T_1 with n_1 vertices and l_1 edges and T_2 with n_2 vertices and l_2 edges.
     - **Child**: T with n vertices and l edges.
     - Constructor always preserves property P.
  3. By structural induction, $P(T)$ is true $\forall T \in RBT$.

Checklist for Structural Induction

- *Analogy:* if the first ancestors had blue eyes, and blue eyes are inherited from one generation to the next, then all of society will have blue eyes.
- You have a recursively defined set $\mathcal{S}$
- You want to prove a property P for all members of $\mathcal{S}$
- Does the property P hold for the base cases?
- Is the property P *preserved* by all the constructor rules?
- Structural induction is **not** how to prove all objects with property P are in $\mathcal{S}$