Strong Induction: Strengthening Induction

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 6

Overview

- Solving harder problems with induction
  - Proving $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
- Strengthening the induction hypothesis
  - Proving $n^2 < 2^n$
  - L-tiling
- Many flavors of induction
  - Leaping Induction
    - Postage
    - $n^3 < 2^n$
  - Strong induction
    - Fundamental Theorem of Arithmetic
    - Games of Strategy

A Hard Problem: $\sum_{i=1}^{n}\frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

Rensselaer

- *Proof.* $P(n)$: $\sum_{i=1}^{n}\frac{1}{\sqrt{i}} \leq 2\sqrt{n}$

1. [**Base case**] $P(1)$ claims that $1 \leq 2$, which is T
2. [**Induction step**] Show $P(n) \to P(n+1)$ for all $n \geq 1$. Direct proof.
   - Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n}\frac{1}{\sqrt{i}} \leq 2\sqrt{n}$
   - Show $P(n+1)$ is T:

$$\sum_{i=1}^{n+1}\frac{1}{\sqrt{i}} \leq 2\sqrt{n+1}$$

$$\sum_{i=1}^{n+1}\frac{1}{\sqrt{i}} = \sum_{i=1}^{n}\frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \qquad \text{[key step]}$$

$$\leq 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \qquad \text{[induction hypothesis]}$$

   - Hm, now what??
   - *Lemma*: $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$

Lemma: $2\sqrt{n} + \frac{1}{\sqrt{n+1}} \leq 2\sqrt{n+1}$

Rensselaer

- Proof. By contradiction.
  - Assume

$$2\sqrt{n} + \frac{1}{\sqrt{n+1}} > 2\sqrt{n+1}$$

  - It follows that (by multiplying by $\sqrt{n+1}$)

$$2\sqrt{n(n+1)} + 1 > 2(n+1)$$
$$2\sqrt{n(n+1)} > 2n + 1$$
$$4n(n+1) > (2n+1)^2$$
$$4n^2 + 4n > 4n^2 + 4n + 1$$
$$0 > 1$$

  - Contradiction!

A Hard Problem: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$

- *Proof.* $P(n)$: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$

1. [**Base case**] $P(1)$ claims that $1 \le 2$, which is T
2. [**Induction step**] Show $P(n) \to P(n+1)$ for all $n \ge 1$. Direct proof.
   - Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} \le 2\sqrt{n}$
   - Show $P(n+1)$ is T: $\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} \le 2\sqrt{n+1}$

$$\sum_{i=1}^{n+1} \frac{1}{\sqrt{i}} = \sum_{i=1}^{n} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{n+1}} \quad \text{[key step]}$$

$$\le 2\sqrt{n} + \frac{1}{\sqrt{n+1}} \quad \text{[induction hypothesis]}$$

$$\le 2\sqrt{n+1} \quad \text{[Lemma]}$$

   - So, $P(n) \to P(n+1)$
3. By induction, $P(n)$ is T $\forall n \ge 1$.

Proving Stronger Claims

- Prove that $n^2 \leq 2^n$ for $n \geq 4$
- *Proof attempt*. [By induction]
- [**Base case**] $P(4)$ claims that $16 \leq 16$, which is T
- [**Induction step**] Assume $P(n)$ is T: $n^2 \leq 2^n$ for $n \geq 4$
  - Need to show $P(n) \rightarrow P(n+1)$:
    $$(n+1)^2 \leq 2^{n+1}$$
  - Note that $(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1$
  - If only we could show $2n + 1 \leq 2^n$
    - Then $2^n + 2n + 1 \leq 2^n + 2^n = 2^{n+1}$
  - With induction, it can be easier to prove a stronger claim.

Strengthen the claim: $Q(n)$ Implies $P(n)$

Rensselaer

- Consider a new claim $Q(n)$: $(i)\ n^2 \le 2^n$ AND $(ii)\ 2n + 1 \le 2^n$
- *Proof*. [By induction]

1. [**Base case**] $Q(4)$ claims $16 \le 16$ AND $9 \le 16$; both are T
2. [**Induction step**] Show $Q(n) \to Q(n+1)$ for $n \ge 4$. Direct proof
   - Assume $Q(n)$ is T: $(i)\ n^2 \le 2^n$ AND $(ii)\ 2n + 1 \le 2^n$
   - Show $Q(n+1)$ is T:

     $$(i)\ (n+1)^2 \le 2^{(n+1)} \text{ AND } (ii)\ 2(n+1) + 1 \le 2^{(n+1)}$$

     $$(i)\!: (n+1)^2 = n^2 + 2n + 1$$
     $$\le 2^n + 2n + 1 \le 2^n + 2^n = 2^{n+1}$$

     - (From the induction hypothesis: $n^2 \le 2^n$ AND $2n + 1 \le 2^n$)

     $$(ii)\!: 2(n+1) + 1 = 2 + 2n + 1$$
     $$\le 2^n + 2^n = 2^{n+1}$$

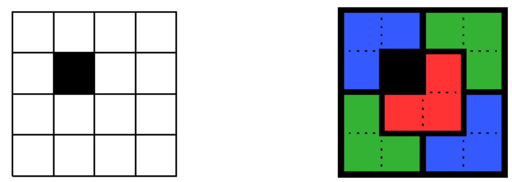
     - (Because $2 \le 2^n$ and $2n + 1 \le 2^n$ from the induction hypothesis)
   - So $Q(n+1)$ is T
3. By induction, $Q(n)$ is T for $n \ge 4$

L-Tile Land

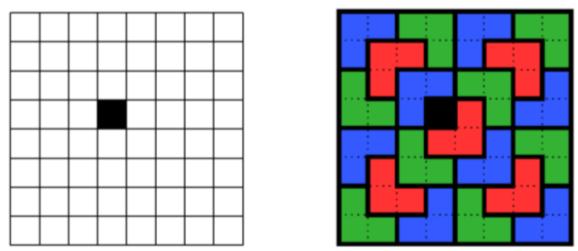
Rensselaer

- Can you tile a $2^n \times 2^n$ patio missing a center square (there's a pot there!). You only have L-shaped tiles
- TINKER!
  - when $n = 1$

  - when $n = 2$



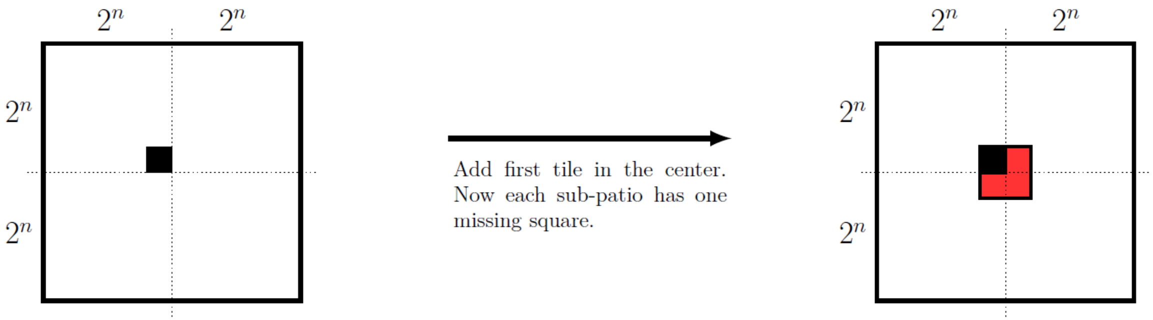
  - when $n = 3$



- $P(n)$: The $2^n \times 2^n$ grid minus a center-square can be L-tiled.

L-Tile Land: Induction Idea

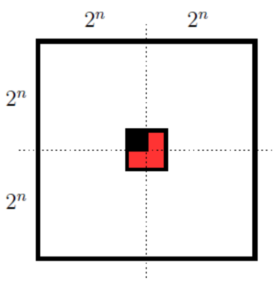
- Suppose $P(n)$ is T. What about $P(n+1)$?
- The $2^{n+1} \times 2^{n+1}$ patio can be decomposed into four $2^n \times 2^n$ patios



- **Problem.** Corner squares are missing. $P(n)$ can be used only if center-square is missing.
- **Solution.** Strengthen claim to also include patios missing corner-squares. $Q(n)$:
  - (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; AND
  - (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled

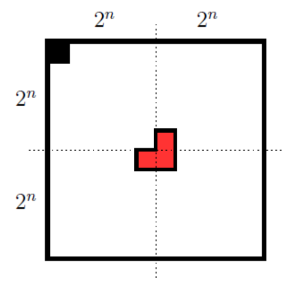
L-Tile Land: Induction Proof of Stronger Claim

- Assume $Q(n)$:
  - (i) The $2^n \times 2^n$ grid missing a **center-square** can be L-tiled; and
  - (ii) The $2^n \times 2^n$ grid missing a **corner-square** can be L-tiled
- Induction step: Must prove two things for $Q(n+1)$, namely (i) *and* (ii).
  - (i) Center square missing



  - use $Q(n)$ with center squares

(ii) Corner square missing



use $Q(n)$ with corner squares

- **Exercise:** Add base cases and complete the formal proof.
- **Exercise 6.4.** What if the missing square is some random square?
  - Strengthen further.

A Tricky Induction Problem

Rensselaer

- Prove $P(n): n^3 < 2^n$, for all $n \geq 10$
- *Proof attempt*. [By induction]
  - [**Base case**] $P(10)$ claims $1000 = 10^3 < 2^{10} = 1024$.
    - True.
  - [**Induction step**] Assume $P(n)$ is T: $n^3 < 2^n$ for $n \geq 10$.
    - Need to show $P(n+1)$ is T:
      $$(n+1)^3 < 2^{n+1}$$
      - Seems hard
    - Consider $P(n+2)$: $(n+2)^3 < 2^{n+2}$?
      $$(n+2)^3 = n^3 + \mathbf{6}n^2 + \mathbf{12}n + \mathbf{8}$$
      $$< n^3 + \boldsymbol{n} \cdot n^2 + \boldsymbol{n^2} \cdot n + \boldsymbol{n^3}$$
        - » (Because $n \geq 10 \rightarrow 6 < n, 12 < n^2, 8 < n^3$)
      $$(n+2)^3 < n^3 + \boldsymbol{n} \cdot n^2 + \boldsymbol{n^2} \cdot n + \boldsymbol{n^3} = 4n^3$$
      $$< 4 \cdot 2^n = 2^{n+2}$$
        - » (From induction hypothesis: $P(n): n^3 < 2^n$)
      - i.e., $P(n) \rightarrow P(n+2)$
    - Not quite induction yet. What can we do?

A Tricky Induction Problem, cont'd

Rensselaer

- Prove $P(n): n^3 < 2^n$, for all $n \geq 10$
- *Proof*. [By induction]

1. [**Base cases**] $P(10)$ claims $1000 = 10^3 < 2^{10} = 1024.$
   $P(11)$ claims $1331 = 11^3 < 2^{11} = 2048.$
   - Both are T.
2. [**Induction step**] Assume $P(n)$ is T: $n^3 < 2^n$ for $n \geq 10.$
   - Need to show $P(n) \rightarrow P(n+2)$: $(n+2)^3 < 2^{n+2}$
     - Consider $P(n+2): (n+2)^3 < 2^{n+2}$?
       $(n+2)^3 = n^3 + \mathbf{6}n^2 + \mathbf{12}n + \mathbf{8}$
       $< n^3 + \boldsymbol{n} \cdot n^2 + \boldsymbol{n^2} \cdot n + \boldsymbol{n^3}$
         » (Because $n \geq 10 \rightarrow 6 < n, 12 < n^2, 8 < n^3$)
       $(n+2)^3 < n^3 + \boldsymbol{n} \cdot n^2 + \boldsymbol{n^2} \cdot n + \boldsymbol{n^3} = 4n^3 < 4 \cdot 2^n = 2^{n+2}$
         » (From induction hypothesis: $P(n): n^3 < 2^n$)
       – i.e., $P(n) \rightarrow P(n+2)$
3. By induction, $P(n+2)$ is T for all $n \geq 10$
   - Already showed $P(10)$ and $P(11)$ are T.

Leaping Induction

Rensselaer

- **Induction**. One base case.

$$\boldsymbol{P(1)} \to P(2) \to P(3) \to \cdots$$

- **Leaping Induction**. More than one base case.

$$\boldsymbol{P(1)} \to P(3) \to P(5) \to \cdots$$
$$\boldsymbol{P(2)} \to P(4) \to P(6) \to \cdots$$

- **Example**. Postage greater than 5¢ can be made using 3¢ and 4¢ stamps.

3¢	4¢	5¢	**6¢**	**7¢**	**8¢**	9¢	10¢	11¢
3	4	-	**3,3**	**3,4**	**4,4**	3,3,3	3,4,3	4,4,3

- $P(n)$: Postage of n¢ can be made using only 3¢ and 4¢ stamps.

$$P(n) \to P(n+3) \text{ (add a 3¢ stamp to } n\text{)}$$

- **Practice**. Exercise 6.6

Fundamental Theorem of Arithmetic

Rensselaer

- The fundamental theorem of arithmetic states that
$$2024 = 2 \times 2 \times 2 \times 11 \times 23$$
  - Huh?
  - Well, it says more than that ☺
- *Theorem [The primes ($\mathcal{P} = \{2,3,5,7,11,13, \ldots\}$) are the atom numbers]*. Suppose $n \geq 2$ is natural number. Then:
  - (i) n **can be written as a product of factors all of which are prime.**
  - (ii) The representation of n as a product of primes is unique (up to reordering).
- What is $P(n)$?
$$P(n)\text{: } n \text{ is a product of primes}$$
- What is the first thing we do?
  - TINKER!

Fundamental Theorem of Arithmetic

Rensselaer

- The prime-factor decomposition of 2024 is:
$$2024 = 2 \times 2 \times 2 \times 11 \times 23$$
- *Theorem*. [The primes ($\mathcal{P} = \{2,3,5,7,11,13,\ldots\}$) are the atom numbers]. Suppose $n \geq 2$. Then:
  – (i) n **can be written as a product of factors all of which are prime.**
  – (ii) The representation of n as a product of primes is unique (up to reordering).
- What is $P(n)$?
$$P(n){:}\ n \text{ is a product of primes}$$
- What is the prime-factor decomposition of 2025:
$$2025 = 5 \times 5 \times 3 \times 3 \times 3 \times 3$$
- Wow! No similarity between the factors of 2024 and 2025
  – **How will** $P(n)$ **help us to prove** $P(n+1)$**?**

Much "Stronger" Induction Claim

Rensselaer

- Do smaller values of n help with 2025?
  - Yes, $2025 = 25 \times 81$
$$P(25) \land P(81) \rightarrow P(2025)$$
  - (like leaping induction)
- **Much Stronger Claim:**
  - $Q(n)$: $2, 3, \ldots, n$ are all products of primes.
  - Compare with: $P(n)$: n is a product of primes
$$Q(n) = P(2) \land P(3) \land P(4) \land \cdots \land P(n)$$
- **Surprise!** The much stronger claim is *much* easier to prove.
  - Also, $Q(n) \rightarrow P(n)$

Fundamental Theorem of Arithmetic: Proof of (i)

- Recall $P(n)$: n is product of primes.
  - Recall $Q(n) = P(2) \wedge P(3) \wedge \cdots \wedge P(n)$
- *Proof*. [By induction that $Q(n)$ is T for all $n \geq 2$.]

1. [**Base case**]. $Q(1)$ claims that 2 is product of primes. True.
2. [**Induction step**] Show that $Q(n) \rightarrow Q(n+1)$ for $n \geq 2$. Direct proof.
   - Assume $Q(n)$ is T: each of $2,3,\ldots,n$ are products of primes
   - Show $Q(n+1)$ is T: each of $2,3,\ldots,n,n+1$ are products of primes
   - Since we assumed $Q(n)$, we know $2,3,\ldots,n$ are products of primes
   - **To prove $Q(n+1)$, we only need to prove $n+1$ is a product of primes!**

Fundamental Theorem of Arithmetic: Proof of (i)

Rensselaer

- *Proof*. [By induction that $Q(n)$ is T for all $n \geq 2$.]

1. [**Base case**]. $Q(1)$ claims that 2 is product of primes. True.
2. [**Induction step**] Show that $Q(n) \rightarrow Q(n+1)$ for $n \geq 2$. Direct proof.
   - Assume $Q(n)$ is T: each of $2,3,\ldots,n$ are products of primes
   - Show $Q(n+1)$ is T: each of $2,3,\ldots,n,n+1$ are products of primes
   - Since we assumed $Q(n)$, we know $2,3,\ldots,n$ are products of primes
   - **To prove $Q(n+1)$, we only need to prove $n+1$ is a product of primes!**
     - Case 1: $n+1$ is prime.
       - Done, nothing to prove.
     - Case 2: $n+1$ is not prime,
       - i.e., $n+1 = kl$, where $2 \leq k, l \leq n$.
       - What now?
         - » Use induction hypothesis!

           $P(k)$: k is product of primes; $P(l)$: l is product of primes.
       - i.e., $n+1 = kl$ is a product of primes and $Q(n+1)$ is T
3. By induction, $Q(n)$ is T, $\forall n \geq 2$.

Strong Induction

- **Strong Induction.** To prove $P(n)\ \forall n \geq 1$ by strong induction, you use induction to prove the *stronger* claim:
  - $Q(n)$: each of $P(1), P(2), \ldots, P(n)$ are T
- Ordinary induction
  - Base case: Prove $P(1)$
  - Induction step: Assume $P(n)$ and prove $P(n+1)$
- **Strong induction**
  - Base case: Prove $Q(1) = P(1)$
  - Induction step: Assume $Q(n) = P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(n)$ and prove $P(n+1)$
- **Strong induction is always easier**

Every $n \geq 1$ has a binary expansion

Rensselaer

- What is $P(n)$ more precisely?
  - $P(n)$: Every $n \geq 1$ is a sum of distinct powers of 2 (its binary expansion)
  - E.g., what is the binary expansion of 22?

$$22 = 2^4 + 2^2 + 2^1 \; (22_{binary} = 10110)$$

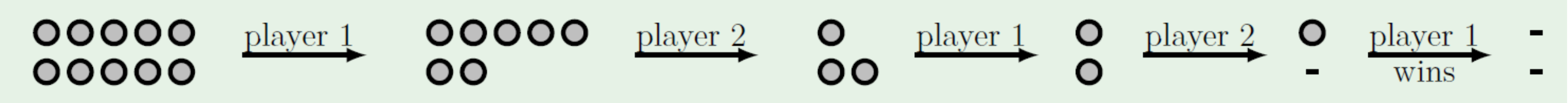
Every $n \geq 1$ has a binary expansion

- *Proof Sketch*.
- [**Base case**] $P(1)$ is T: $1 = 2^0$
- [**Induction step**] Assume $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ and prove $P(n+1)$
  - If n is even, then
  $$n + 1 = 2^0 + n_{binary}$$
    - e.g., $23 = 2^4 + 2^2 + 2^1 + 2^0$
  - If n is odd, then multiply each term in the expansion of $\frac{1}{2}(n+1)$ by 2
    - This gets us $n + 1$
    - e.g., $24 = 2 \times 12_{binary} = 2 \times (2^3 + 2^2) = 2^4 + 2^3$
  - Why does $\frac{1}{2}(n+1)$ have an expansion?
    - Strong induction!
- **Exercise.** Give the formal proof by strong induction.

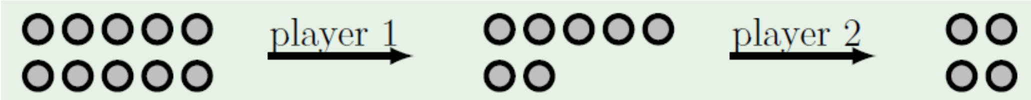
Applications of Induction

Rensselaer

- Greedy or recursive algorithms, games of strategy
- Consider the game of Equal Pile Nim (old English/German: to steal or pilfer)
  - two players take turns taking pennies from two equal rows of pennies
  - each player can take an arbitrary number of pennies from one row
  - the player to take the last stone wins



- Claim: $P(n)$: Player 2 can win the game that starts with n pennies per row.
  - Equalization strategy:



  - Player 2 can always return the game to *smaller* equal piles.
  - If Player 2 wins the smaller game, Player 2 wins the larger game.
    - That's strong induction!
- **Exercise.** Give the full formal proof by strong induction.
- **Challenge.** What about more than 2 piles? What about unequal piles? (Problem 6.20).

Investigate Further in the Problems

Rensselaer

- Uniqueness of binary representation as a sum of distinct powers of 2:
  - **Problem 6.27**
- General Nim:
  - **Problem 6.39**

Checklist When Approaching an Induction Problem

Rensselaer

- Are you trying to prove a "For all . . . " claim?
- Identify the claim $P(n)$, especially the parameter n. Here is an example.
  - Prove: $geometric\ mean \leq arithmetic\ mean$. What is $P(n)$? What is n?
  - $P(n)$: $geometric\ mean \leq arithmetic\ mean$ for every set of $n > 0$ numbers
  - **Identifying the right claim is important.**
    You may fail because you try to prove too much. Your $P(n+1)$ is too heavy a burden. You may fail because you try to prove too *little*. Your $P(n)$ is too weak a support. You must balance the strength of your claim so that the support is just enough for the burden. —G. Polya (paraphrased).
- Tinker. Does the claim hold for small n ($n = 1, 2, 3, \ldots$)? These become base cases.
- Tinker. Can you see why (say) $P(5)$ follows from $P(1), P(2), P(3), P(4)$?
  - This is the crux of induction; to build up from smaller n to a larger n.
- Determine the type of induction: try strong induction first.
- Write out the skeleton of the proof to see exactly what you need to prove.
- Determine and prove the base cases.
- Prove $P(n+1)$ in the induction step. You *must* use the induction hypothesis.