Induction: Proving “For All…”

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 5

Overview: Induction, Proving "…for all…"

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- What is induction
- Why do we need it?
- The principle of induction
  - Toppling the dominos
  - The induction template
- Examples
- Induction, Well-Ordering and the Smallest Counter-Example

Dispensing postage using 5¢ and 7¢ stamps

- How do I pay for a 19¢ letter?
  - using 7,7,5 stamps
- How about 20?
  - using 5,5,5,5 stamps
- How about 21?
  - using 7,7,7 stamps
- How about 22?
  - using 5,5,5,7 stamps
- How about 23?
  - ?
  - Looks hard

Dispensing postage using 5¢ and 7¢ stamps

- How do I pay for a 19¢ letter?
  - using 7,7,5 stamps
  - What about 24?
    - 7,7,5,5
- How about 22?
  - using 5,5,5,7 stamps
  - What about 27?
    - 5,5,5,7,5
- Can every postage greater than 23¢ can be dispensed?
  - Intuitively, yes
  - Induction formalizes this intuition

Why do we need induction?

- Predicate: $P(n)$ = "5¢ and 7¢ stamps can make postage n."
  - Claim: $\forall n \geq 24 \colon P(n)$
  - Seems true
- Predicate: $P(n)$ = "$n^2 - n + 41$ a prime number."
  - Claim: $\forall n \geq 1 \colon P(n)$
  - Try different n
    - $n = 1$: 41 (prime)
    - $n = 2$: 43 (prime)
    - $n = 3$: 47 (prime)
    - $n = 4$: 53 (prime)
    - …
    - $n = 41$: 1681 (not prime!)

Why do we need induction?, cont'd

- Predicate: $P(n)$ = "$4n - 1$ is divisible by 3."
  - Claim: $\forall n \geq 1: P(n)$
  - Try different n
    - $n = 1 \rightarrow 4 - 1 = 3$ (yes)
    - $n = 2 \rightarrow 8 - 1 = 7$ (nope)
- How can we prove something for *all* $n \geq 1$? Checking each n takes too long!
- Prove for general n. Can be tricky.
- **Induction.** Systematic.

Is $4^n - 1$ divisible by 3 for $n \geq 1$?

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- Predicate: $P(n)$ = "$4^n - 1$ is divisible by 3."
- We proved
  - IF $4^n - 1$ is divisible by 3, THEN $4^{n+1} - 1$ is divisible by 3

- *Theorem*: Let x be any real number, i.e., $x \in \mathbb{R}$. IF $4^x - 1$ is divisible by 3, THEN $4^{x+1} - 1$ is divisible by 3.
- *Proof*: We prove the claim using a direct proof.
  1. Assume that p is T, that is $4^x - 1$ is divisible by 3.
  2. This means that $4^x - 1 = 3k$ for an integer k, or that $4^x = 3k + 1$
  3. Observe that $4^{x+1} = 4 \times 4^x$. Using $4^x = 3k + 1$,
     note that $4^{x+1} = 4(3k + 1) = 12k + 4$
  4. Therefore $4^{x+1} - 1 = 12k + 3 = 3(4k + 1)$ is a multiple of 3 ($4k + 1$ is an integer)
  5. Since $4^{x+1} - 1$ is a multiple of 3, we have shown that $4^{x+1} - 1$ is divisible by 3
  6. Therefore, the statement claimed in q is T

Is $4^n - 1$ divisible by 3 for $n \geq 1$?

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- Predicate: $P(n)$ = "$4^n - 1$ is divisible by 3."
- We proved
  - IF $4^n - 1$ is divisible by 3, THEN $4^{n+1} - 1$ is divisible by 3
  - So we proved: IF $P(n)$ THEN $P(n+1)$
  - i.e., $P(n) \rightarrow P(n+1)$
- What use is this?
  - (Reasoning in the absence of facts.)

Is $4^n - 1$ divisible by 3 for $n \geq 1$?

- We proved
  - IF $4^n - 1$ is divisible by 3, THEN $4^{n+1} - 1$ is divisible by 3
  - So we proved IF $P(n)$ THEN $P(n+1)$
  - i.e., $P(n) \rightarrow P(n+1)$
- From tinkering, we know

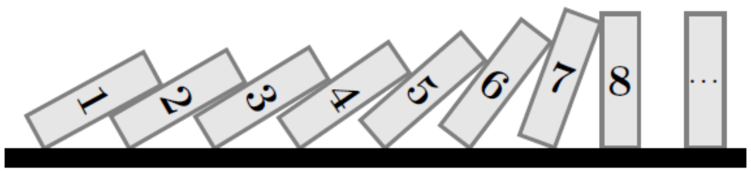
  $P(1)$ is T: $4^1 - 1 = 3$ is divisible by 3
- $P(1) \rightarrow P(2)$
- $P(2) \rightarrow P(3)$
- $P(3) \rightarrow P(4)$
- $P(4) \rightarrow P(5)$
- When does this end??

Is $4^n - 1$ divisible by 3 for $n \geq 1$?

- We know $P(1)$ is T

  $4^1 - 1 = 3$ is divisible by 3

- We also know $P(n) \rightarrow P(n+1)$
- By induction, $P(n)$ is T for all $n \geq 1$



- $P(n)$ form an infinite chain of dominos
- Topple the first and they *all* fall
- **Practice**: Exercise 5.2

Induction Template

- **Induction to prove**: $\forall n \geq 1: P(n)$
- *Proof*. We use induction to prove: $\forall n \geq 1: P(n)$
  1. Show that $P(1)$ is T ("simple" verification) **[base case]**
  2. Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$ **[induction step]**
     - Use <u>Direct</u> proof
     - Assume $P(n)$ is T
       – (valid derivations)
       – must show for any $n \geq 1$
       – must use $P(n)$ here
     - Show $P(n+1)$ is T

Induction Template

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- **Induction to prove**: $\forall n \geq 1: P(n)$
- *Proof*. We use induction to prove: $\forall n \geq 1: P(n)$
  1. Show that $P(1)$ is T ("simple" verification) **[base case]**
  2. Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$ **[induction step]**
     - Use Proof by <u>Contraposition</u>
     - Assume $P(n+1)$ is F
       - (valid derivations)
       - must show for any $n \geq 1$
       - must use $\neg P(n+1)$ here
     - Show $P(n)$ is F

Induction Template

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- **Induction to prove**: $\forall n \geq 1 \colon P(n)$
- *Proof*. We use induction to prove: $\forall n \geq 1 \colon P(n)$
  1. Show that $P(1)$ is T ("simple" verification) **[base case]**
  2. Show $P(n) \rightarrow P(n+1)$ for $n \geq 1$ **[induction step]**

     - Use <u>Direct</u> proof
     - Assume $P(n)$ is T
       - (valid derivations)
       - must show for any $n \geq 1$
       - must use $P(n)$ here
     - Show $P(n+1)$ is T

     - Use Proof by <u>Contraposition</u>
     - Assume $P(n+1)$ is F
       - (valid derivations)
       - must show for any $n \geq 1$
       - must use $\neg P(n+1)$ here
     - Show $P(n)$ is F

  3. Conclude: by induction: $\forall n \geq 1 \colon P(n)$

Induction Template, cont'd

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- Prove the implication $P(n) \rightarrow P(n+1)$ for a general $n \geq 1$
- Why is this easier than just proving $P(n)$ for general n?
  - Assuming $P(n)$ is T gives us a lot of information to work with
- Assume $P(n)$ is T, and reformulate it mathematically
- Somewhere in the proof you *must* use $P(n)$ to prove $P(n+1)$
- End with a statement that $P(n+1)$ is T

Sum of Integers

- What is the sum $1 + 2 + 3 + \cdots + (n - 1) + n$
  - Can you give an expression as a function of n?
- The mathematician Gauss was one day sitting in class and was bored, so his teacher asked him to calculate $1 + 2 + \cdots + 100$
  - he started playing around with numbers:

$$S(n) = 1 + 2 + \cdots + n$$
$$S(n) = n + n - 1 + \cdots + 1$$

- So, $2S(n) = (n + 1) + (n + 1) + \cdots + (n + 1)$
  - i.e., $2S(n) = n \times (n + 1)$
  - i.e., $S(n) = \frac{n(n+1)}{2}$
- This is direct proof!
  - Note that this proof technique requires ingenuity in general

Proof by induction: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

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- *Proof*: (By induction) $P(n)$: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$

1. [**Base Case**] $P(1)$ claims that $1 = \frac{1}{2} \times 1 \times (1+1)$
   - Clearly T
2. [**Induction Step**] We show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$, using direct proof.
   - Assume (induction hypothesis) $P(n)$ is T: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$
   - Need to show $P(n+1)$ is T: $\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+1+1)$

$$\sum_{i=1}^{n+1} i = \left(\sum_{i=1}^{n} i\right) + (n+1) \qquad \textbf{[key step]}$$

$$= \frac{1}{2}n(n+1) + (n+1) \qquad \textbf{[induction hypothesis } \boldsymbol{P(n)}\textbf{]}$$

$$= (n+1)\left(\frac{1}{2}n+1\right) = \frac{1}{2}(n+1)(n+2) \qquad \textbf{[algebra]}$$

$$= \frac{1}{2}(n+1)(n+2) \qquad \textbf{[what needed to be shown]}$$

3. By induction, $P(n)$ is T for all $n \geq 1$

BEWARE of going in the wrong direction!

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- If we had started from $n + 1$

$$\sum_{i=1}^{n+1} i = \frac{1}{2}(n+1)(n+2)$$

  - This is what we would like to show

- It follows that:

$$\sum_{i=1}^{n+1} i - (n+1) = \frac{1}{2}(n+1)(n+2) - (n+1)$$

$$\sum_{i=1}^{n} i = (n+1)\left(\frac{1}{2}n + 1 - 1\right)$$

$$= \frac{1}{2}n(n+1)$$

Hoooray!

Or is it…

BEWARE of going in the wrong direction!

- Suppose we assume $7 = 4$
- This means that $4 = 7$
  - because $(a = b) \to (b = a)$
- If we add both equations, we get $11 = 11$
  - Just because the final result makes sense doesn't mean that we did something right
  - By assuming $4 = 7$, we proved that $11 = 11$
  - But did we actually prove $4 = 7$?
- To start, you can **NEVER** assert (as though it's true) what you are trying to prove

Sum of Integer Squares

- What is the sum $1^2 + 2^2 + 3^2 + \cdots + (n-1)^2 + n^2$?
  - Need to channel our inner Gauss
  - Unfortunately, he didn't solve this one
    - Or didn't think it was important enough to write down…
- Let's play around with some numbers first

$$S(1) = 1$$
$$S(2) = 5$$
$$S(3) = 14$$
$$S(4) = 30$$
$$S(5) = 55$$
$$S(6) = 91$$
$$S(7) = 140$$

Sum of Integer Squares, cont'd

- Let's play around with some numbers first

  $S(1) = 1, S(2) = 5, S(3) = 14, S(4) = 30, S(5) = 55, S(6) = 91, S(7) = 140$
- How about $S'(n) = S(n+1) - S(n)$?
  - All the squares: 4,9,16,25,36,49
- How about $S''(n) = S'(n+1) - S'(n)$?
  - All odd numbers: 5,7,9,11,13
- How about $S'''(n) = S''(n+1) - S''(n)$?
  - Constant: 2,2,2,2
- Hm… Difference is kind of like a derivative
  - If a function's 4th derivative is 0, then we know the function is a 3rd order polynomial
  - Perhaps we can approximate $S(n)$ in a Taylor-series-like way and see if that works

Sum of Integer Squares, cont'd

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- Recall Taylor series expansion (around point x_0):

$$\hat{f}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \frac{1}{6}f'''(x_0)(x - x_0)^3$$

  - Higher-order terms are 0 if third derivative is constant
- "Taylor series" guess

$$S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3$$

- Let's plug in a few values for n and see if can solve for the a's

$$S(1) = 1 = a_0 + a_1 + a_2 + a_3$$
$$S(2) = 5 = a_0 + 2a_1 + 4a_2 + 8a_3$$
$$S(3) = 14 = a_0 + 3a_1 + 9a_2 + 27a_3$$
$$S(4) = 30 = a_0 + 4a_1 + 16a_2 + 64a_3$$

  - How do we solve a linear system of equations?
  - Gaussian elimination! (thank you, Gauss, after all)

Sum of Integer Squares, cont'd

- "Taylor series" guess

$$S(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3$$

- Let's plug in a few values for n and see if can solve for the a's

$$S(1) = 1 = a_0 + a_1 + a_2 + a_3$$
$$S(2) = 5 = a_0 + 2a_1 + 4a_2 + 8a_3$$
$$S(3) = 14 = a_0 + 3a_1 + 9a_2 + 27a_3$$
$$S(4) = 30 = a_0 + 4a_1 + 16a_2 + 64a_3$$

- Solution is $a_0 = 0, a_1 = \frac{1}{6}, a_2 = \frac{1}{2}, a_3 = \frac{1}{3}$
  - Solve for a_0 in terms of other a's; then solve for a_1, etc.
- So guess is $S(n) = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$
  - $S(1) = 1, S(2) = 5, S(3) = 14, \ldots$
  - Hm, seems correct. Let's prove it using induction!

Proof: $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

- *Proof*: (By induction)

$$P(n): \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$$

1. [**Base case**] $P(1)$ claims that $1 = \frac{1}{6} \times 1 \times 2 \times 3$, which is T
2. [**Induction step**] Show $P(n) \to P(n+1)$ for all $n \geq 1$. Direct proof.
   - Need to show $P(n+1)$ is T:

$$\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

$\sum_{i=1}^{n+1} i^2 = \left(\sum_{i=1}^{n} i^2\right) + (n+1)^2$ [**key step**]

$= \frac{1}{6}n(n+1)(2n+1) + (n+1)^2$ [**induction hypothesis $P(n)$**]

$= \frac{1}{6}(n+1)\left(2n^2 + 7n + 6\right)$ [**algebra**]

$= \frac{1}{6}(n+1)\left(2n^2 + 4n + 3n + 6\right)$ [**algebra**]

$= \frac{1}{6}(n+1)(n+2)(2n+3)$ [**what needed to be shown**]

Proof: $S(n) = \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3$

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- *Proof*: (By induction)

$$P(n): \sum_{i=1}^{n} i^2 = \frac{1}{6}n + \frac{1}{2}n^2 + \frac{1}{3}n^3 = \frac{1}{6}n(n+1)(2n+1)$$

1. [**Base case**] $P(1)$ claims that $1 = \frac{1}{6} \times 1 \times 2 \times 3$, which is T
2. [**Induction step**] Show $P(n) \rightarrow P(n+1)$ for all $n \geq 1$. Direct proof.
   - Need to show $P(n+1)$ is T:

$$\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

$\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)\left(2n^2 + 4n + 3n + 6\right)$ [**algebra**]

$= \frac{1}{6}(n+1)(n+2)(2n+3)$ [**what needed to be shown**]

   - So $P(n+1)$ is T

- By induction, $P(n)$ is T for all $n \geq 1$

Induction gone wrong

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- Suppose we proved $P(1) \to P(2) \to P(3) \to P(4) \to \cdots$
- What is missing?
  - **No base case!**
- Remember, $F \to T$!
- Suppose we want to prove:

$$P(n): n \geq n + 1 \text{ for all } n \geq 1$$

  - Add 1 to both sides:

$$n \geq n + 1 \to n + 1 \geq n + 2$$

  - Therefore, $P(n) \to P(n+1)$
- [Every link is proved, but without the base case, you have *nothing*.]
  - **Broken chain!**

Induction gone wrong, cont'd

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- False: $P(n)$: "all balls in any set of n balls are the same color."
  - **Base case.** $P(1)$ is T because there is only 1 ball
  - **Induction step.** Suppose any set of n balls have the same color.
    - Consider any set of $n + 1$ balls $b_1, b_2, \dots, b_n, b_{n+1}$.
    - So, $b_1, b_2, \dots, b_n$ have the same color and $b_2, \dots, b_n, b_{n+1}$ have the same color.
    - Thus $b_1, b_2, \dots, b_n, b_{n+1}$ have the same color.
  - Does that mean $P(n) \rightarrow P(n + 1)$ for all $n \geq 1$?
    - Well, $P(1) \rightarrow P(2)$ is F
- [A *single* broken link kills the entire proof.]
- How would you "fix" this proof?
  - Need two base cases!
  - Of course, now we can't prove $P(2)$!
  - Phew, what a relief – the world is colorful after all!

Well Ordering Principle

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- Recall the **Well-ordering Principle**:
  - *Any* non-empty set of natural numbers has a minimum element.
- Induction follows from well ordering
  - Let $P(1)$ and $P(n) \rightarrow P(n+1)$ be T
- Suppose $P(n_*)$ fails for the **smallest** counter-example n_* (well-ordering).

$$P(1) \rightarrow P(2) \rightarrow P(3) \rightarrow \cdots \rightarrow P(n_* - 1) \rightarrow P(n_*) \rightarrow \cdots$$

  - Now how can $P(n_* - 1) \rightarrow P(n_*)$ be T?
- **Any induction proof can also be done using well-ordering.**

Example Well-ordering Proof: $n < 2^n$ for all $n \geq 1$

- First prove it with induction
- Proof: [Induction] $P(n)$: $n < 2^n$

1. [**Base case**] $P(1)$ claims that $1 < 2^1$, which is T
2. [**Induction step**] Assume $P(n)$ is T: $n < 2^n$
   – Need to show $P(n+1)$ is T:
   $$n + 1 < 2^{n+1}$$
   – From the induction hypothesis:
   $$\begin{aligned} n + 1 &\leq n + n \\ &\leq 2^n + 2^n = 2 \times 2^n = 2^{n+1} \end{aligned}$$

Example Well-ordering Proof: $n < 2^n$ for all $n \geq 1$

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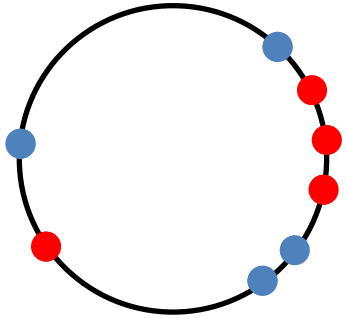
- *Proof*: [Well-ordering] Proof by **contradiction**.
  - Assume that there is an $n \geq 1$ for which $n \geq 2^n$
  - Let n_* be the **minimum** such **counter-example**, $n_* \geq 2^{n_*}$
    - Using the well ordering axiom
  - Since $1 < 2^1$, then $n_* \geq 2$
  - Since $n_* \geq 2$, $\frac{1}{2}n_* \geq 1$ and so,

$$n_* - 1 \geq n_* - \frac{1}{2}n_* = \frac{1}{2}n_*$$
$$\geq \frac{1}{2} \times 2^{n_*} = 2^{n_*-1}$$

- So, $n_* - 1$ is a *smaller* counter example. **FISHY!**
- The **method of minimum counter-example** is very powerful.

Getting Good at Induction

- TINKER, TINKER, TINKER
- PRACTICE, PRACTICE, PRACTICE
- Just because something is not immediately obvious doesn't mean you should give up
- **Challenge.** A circle has $2n$ distinct points, n are red and n are blue.
  - Prove that for all $n \geq 1$, there exists a blue point such that one can start at that blue point and move clockwise always having passed as many blue points as red.



- **Practice.** All exercises and pop-quizzes in chapter 5.
- **Strengthen.** Problems in chapter 5.