Making Precise Statements

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 3

Today

- Making a precise statement: the proposition
- Complicated precise statements: the compound proposition
  - Truth tables
- Claims about many things
  - Predicates
  - Quantifiers
  - Proofs with quantifiers

Statements can be ambiguous

- Precise statements

  $2 + 2 = 4$ (True)

  $2 + 2 = 5$ (False)

- Not-so-precise statements
  - You can have ice cream or cake
    - Can I have both?
    - Exclusive or Inclusive Or?
  - If pigs can fly, then you get an A
    - Pigs can't fly, so do you still get an A?
    - False → Anything
  - There is a room for every student
    - Do all students share the same room?
    - Does each student get an individual room?

Why is ambiguity bad?

- We want to **prove** things!
- Need to know when and if computers implement correct algorithms!
- Beware of ambiguous statements
  - Natural language is ambiguous by design
  - That’s why we have math

Propositions are True (T) or False (F)

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- Propositions are represented using lowercase letters

$$p, q, r, s, \ldots$$

- Piglet can fly
  - False
- You got an A
  - Hmmm.. T?
- 4^2 is even
  - True

- There are actually many types of logics out there
  - E.g., fuzzy logic includes probabilities
  - There are logics that also include a third value, Maybe/Don't know
  - We are only going to focus on classical logic
    - If something is not T, then it must be F

Compound Statements

- Piglet can fly OR 4^2 is even
  - True
- Piglet can fly → You got an A
  - True
  - False → anything
- Piglet cannot fly → You got an A
  - ?
  - Depends on the value of “You got an A”

Notation

- Conjunction

  $p \land q$

  p AND q

- Disjunction

  $p \lor q$

  p OR q

- Negation

  $\neg p$

  NOT p

- Implication

  $p \rightarrow q$

  p IMPLIES q

Negation

- The negation $\neg p$ is F when p is T
- The negation $\neg p$ is T when p is F
- Piglet can fly is F
- $\neg$(Piglet can fly) is T
- Notice how English quickly becomes redundant/ambiguous
  - Piglet cannot fly
  - It is not the case that Piglet can fly

Conjunction

- Both p and q must be T for $p \wedge q$ to be T
  - Otherwise $p \wedge q$ is F
- Piglet can fly AND You got an A
  - F(alse)
  - (Piglet can fly) $\wedge$ (You got an A) = F
- Piglet cannot fly AND You got an A
  - ?
  - Depends on the value of (You got an A)

Disjunction

- Both p and q must be F for $p \vee q$ to be F
  - Otherwise it is T
- (Piglet cannot fly) ∨ (You got an A)
  - Depends on the value of (You got an A)
- ¬(Piglet cannot fly) ∨ (You got an A) = T
  - Why?
  - Because ¬(Piglet cannot fly) = T
- (You can have cake) OR (You can have ice-cream)
  - Can you have both?
  - Yes, this is Inclusive OR
  - Exclusive OR is true when **exactly one** is true

Truth Table

- Essentially a function that maps the value of p and q to the statement we're trying to make
- Defines the meaning of these operators

p	q	$\neg p$	$p \wedge q$	$p \vee q$
F	F	T	F	F
F	T	T	F	T
T	F	F	F	T
T	T	F	T	T

- Can also use this in the case of any logic formula

Implication

- Piglet can fly → You got an A
  - IF Piglet can fly THEN You got an A
- IF n^2 is even, THEN n is even
  - Is every even square the square of an even number?
- IF (it rained last night) THEN (the grass is wet)

  p = (it rained last night)

  q = (the grass is wet)
  - In logic notation: $p \rightarrow q$
  - What does it *mean* for this common-sense implication to be true?
    - We have built a model of the world
    - Whenever we observe p, we can make conclusions about q
    - If we don't observe p, our model tells us nothing about q
    - If only observe q, can't conclude anything about p
  - What can you conclude? Did it rain last night? Is the grass wet?

Implication, cont'd

- IF (it rained last night) THEN (the grass is wet)
  - What does it *mean* for this common-sense implication to be true?
  - What can you conclude? Did it rain last night? Is the grass wet?
- Suppose you look at the weather report for last night, and it indeed rained
- Is the grass wet?
  - YES
- For a true implication $p \rightarrow q$, you can conclude $q = T$ when $p = T$

Implication, cont'd

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- IF (it rained last night) THEN (the grass is wet)
  - What does it *mean* for this common-sense implication to be true?
  - What can you conclude? Did it rain last night? Is the grass wet?
- Suppose you see wet grass in the morning
  - Did it rain?
  - Can't tell
- For a true implication $p \rightarrow q$, when $q = T$ you **cannot** conclude $p = T$

Implication, cont'd

- IF (it rained last night) THEN (the grass is wet)
  - What does it *mean* for this common-sense implication to be true?
  - What can you conclude? Did it rain last night? Is the grass wet?
- Suppose you see dry grass in the morning
  - Did it rain?
  - No
    - Our model of the world assumes the grass MUST BE wet if it rained
- For a true implication $p \rightarrow q$, when q is F, you *can* conclude p is F

Implication, cont'd

- IF (it rained last night) THEN (the grass is wet)
  - What does it *mean* for this common-sense implication to be true?
  - What can you conclude? Did it rain last night? Is the grass wet?
- Suppose you see no rain in the weather report
  - Is the grass wet?
  - Can't tell
- For a true implication $p \rightarrow q$, when p is F, you cannot conclude q is F

Implication: inferences when new information comes

For a **true** implication $p \to q$:

When p is T, you can conclude that q is T.

When q is T, you **cannot** conclude p is T.

When p is F, you **cannot** conclude q is F.

When q is F, you can conclude p is F.

Falsifying IF (it rained last night) THEN (the grass is wet)

- You are a scientist collecting data to *verify* that the implication is valid (true)
- **One night it rained. That morning the grass was dry.**
  - New information
- What do you think about the implication now?
- This is a *falsifying scenario*
  - IF (it rains) THEN (the grass is wet)
  - False
- Our model of the world was wrong
- $p \rightarrow q$ is only F when $p = T$ and $q = F$
  - In all other cases, $p \rightarrow q = T$

Implication is EXTREMELY important

- All these are $p \rightarrow q$ (p = "it rained last night" and q = "the grass is wet"):
  - If it rained last night then the grass is wet (IF p THEN q)
  - It rained last night implies the grass is wet (p IMPLIES q)
  - It rained last night only if the grass is wet (p ONLY IF q)
  - The grass is wet if it rained last night (q IF p)
  - The grass is wet whenever it rains (q WHENEVER p)
- Notice that there are multiple English descriptions the same logical statement

Implication Truth Table

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$\boldsymbol{p \rightarrow q}$
F	F	T	F	F	T
F	T	T	F	T	T
T	F	F	F	T	F
T	T	F	T	T	T

Example

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- IF (you are hungry OR you are thirsty) THEN you visit the cafeteria
- $(p \vee q) \rightarrow r$
  - where $p =$ you are hungry, $q =$ you are thirsty, $r =$ you visit the cafeteria
- You are thirsty: q is T.
  - There are two rows where q is T and $(p \vee q) \rightarrow r$ is T
  - In both cases r is T (you visit the cafeteria)

	p	q	r	$(p \vee q)$	$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r}$
1.	F	F	F	F	T
2.	F	F	T	F	T
3.	F	T	F	T	F
4.	F	T	T	T	T
5.	T	F	F	T	F
6.	T	F	T	T	T
7.	T	T	F	T	F
8.	T	T	T	T	T

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- You are thirsty: q is T.
  - There are two rows where q is T and $(p \vee q) \rightarrow r$ is T
  - In both cases r is T
    (you visit the cafeteria)
- You did visit the cafeteria: r is T.
  - Are you hungry?
    - We don't know.
  - Are you thirsty?
    - We don't know.
  - (You accompanied your hungry friend)

	p	q	r	$(p \vee q)$	$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r}$
1.	F	F	F	F	T
2.	F	F	T	F	T
3.	F	T	F	T	F
4.	F	T	T	T	T
5.	T	F	F	T	F
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7.	T	T	F	T	F
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Example

- IF (you are hungry OR you are thirsty) THEN you visit the cafeteria
- $(p \vee q) \rightarrow r$
  - where $p =$ you are hungry, $q =$ you are thirsty, $r =$ you visit the cafeteria
- You are thirsty: q is T. In both cases r is T (you visit the cafeteria)
- You did visit the cafeteria: r is T.
  - Are you hungry? We don't know.
  - Are you thirsty? We don't know.
  - (You accompanied your hungry friend)

	p	q	r	$(p \vee q)$	$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r}$
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3.	F	T	F	T	F
4.	F	T	T	T	T
5.	T	F	F	T	F
6.	T	F	T	T	T
7.	T	T	F	T	F
8.	T	T	T	T	T

Example

- IF (you are hungry OR you are thirsty) THEN you visit the cafeteria
- $(p \lor q) \rightarrow r$
  - where $p =$ you are hungry, $q =$ you are thirsty, $r =$ you visit the cafeteria
- You are thirsty: q is T.
  - There are two rows where q is T and $(p \lor q) \rightarrow r$ is T
  - In both cases r is T (you visit the cafeteria)
- You did visit the cafeteria: r is T.
  - Are you hungry? We don't know.
  - Are you thirsty? We don't know.
  - (You accompanied your hungry friend)
- You did not visit the cafeteria: r is F
  - p and q are both F
  - (You are neither hungry nor thirsty.)

	p	q	r	$(p \lor q)$	$(\boldsymbol{p} \lor \boldsymbol{q}) \rightarrow \boldsymbol{r}$
1.	F	F	F	F	T
2.	F	F	T	F	T
3.	F	T	F	T	F
4.	F	T	T	T	T
5.	T	F	F	T	F
6.	T	F	T	T	T
7.	T	T	F	T	F
8.	T	T	T	T	T

Example

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- IF (you are hungry OR you are thirsty) THEN you visit the cafeteria
- $(p \vee q) \rightarrow r$
  - where $p =$ you are hungry, $q =$ you are thirsty, $r =$ you visit the cafeteria
- You are thirsty: q is T.
  - There are two rows where q is T and $(p \vee q) \rightarrow r$ is T
  - In both cases r is T (you visit the cafeteria)
- You did visit the cafeteria: r is T.
  - Are you hungry? We don't know.
  - Are you thirsty? We don't know.
  - (You accompanied your hungry friend)
- You did not visit the cafeteria: r is F
  - p and q are both F
  - (You are neither hungry nor thirsty.)

	p	q	r	$(p \vee q)$	$(\boldsymbol{p} \vee \boldsymbol{q}) \rightarrow \boldsymbol{r}$
1.	F	F	F	F	T
2.	F	F	T	F	T
3.	F	T	F	T	F
4.	F	T	T	T	T
5.	T	F	F	T	F
6.	T	F	T	T	T
7.	T	T	F	T	F
8.	T	T	T	T	T

Equivalent Compound Statements

p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$\neg p \vee q$	$q \rightarrow p$
F	F	T	T	T	T
F	T	T	T	T	F
T	F	F	F	F	T
T	T	T	T	T	T
		rains → wet grass	dry grass → no rain	no rain ∨ wet grass	wet grass → rain

$$p \rightarrow q \overset{\text{eqv}}{\equiv} \neg q \rightarrow \neg p \overset{\text{eqv}}{\equiv} \neg p \vee q$$

- **Order is very important!**
  - In particular, $p \rightarrow q$ and $q \rightarrow p$ **do not** mean the same thing!
- IF I'm dead, THEN my eyes are closed **vs.** IF my eyes are closed, THEN I'm dead

Proving an Implication: Reasoning without Facts

- IF (n^2 is even) THEN (n is even)

$p:\ n^2$ is even
$q:\ n$ is even

$p \to q$

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

- What is n? How to prove?
  - We must show that the highlighted row *cannot* occur.
  - i.e., n is odd cannot be the case
- In this row, q is F: $n = 2k + 1$
- $n^2 = (2k+1)^2 = 2(2k^2 + 2k) + 1$
- p cannot be T. This row cannot happen: $p \to q$ is always T

Quantifiers

- Every person has a soulmate
- John has some gray hair
- Everyone has some gray hair
- Any map can be colored with 4 colors with adjacent countries having different colors
- Every even integer $n > 2$ is the sum of 2 primes *(Goldbach conjecture, 1742)*
  - Still not proven, but holds for numbers up to at least 4×10^{18}
- Someone broke this faucet
- There exists a creature with blue eyes and blonde hair
- All cars have four wheels

Quantifiers, etc.

- These statements are more complex because of *quantifiers*:
  - EVERY; A; SOME; ANY; ALL; THERE EXISTS
- Compare:
  - My Tesla has four wheels
  - ALL cars have four wheels

Predicates are like functions

- ALL cars have four wheels
- Define predicate $P(c)$ and its domain
- $C = \{c \mid c \; is \; a \; car\}$
  - set of cars
- $P(c)$ = “car c has four wheels”
- “for all c in C, the statement $P(c)$ is true.”

$$\forall c \in C: P(c)$$

  - (∀ means “for all”)

Predicates are like functions, cont'd

- Predicates

$$P(c) = \text{“car } c \text{ has four wheels”}$$

  - Input: parameter $c \in C$
  - Output: **statement** $P(c)$
  - Example: $P(Jen's\ Car)$ = “car $Jen's\ Car$ has four wheels”

$$\forall c \in C : P(c)$$

    - Meaning: for all $c \in C$, the statement $P(c)$ is T

- Functions:

$$f(x) = x^2$$

  - Input: parameter $x \in \mathbb{R}$
  - Output: **value** $f(x)$
  - Example: $f(5) = 25$

$$\forall x \in \mathbb{R}, f(x) \geq 0$$

  - Meaning: for all $x \in \mathbb{R}$, $f(x)$ is ≥ 0

Example

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- There EXISTS a creature with blue eyes and blonde hair
- Define predicate $Q(a)$ and its *domain*

$$A = \{a | a \text{ is a creature}\}$$

  – set of creatures

$$Q(a) = \text{“}a \text{ has blue eyes and blonde hair”}$$

- “there exists a in A for which the statement $Q(a)$ is true.”

$$\exists a \in A: Q(a)$$

  ∃ means “there exists”

- $G(a)$ = “a has blue eyes”
- $H(a)$ = “a has blonde hair”
- $\exists a \in A: (G(a) \wedge H(a))$

Example, cont'd

- $G(a)$ = "a has blue eyes"
- $H(a)$ = "a has blonde hair"
- $\exists a \in A: (G(a) \land H(a))$
  - compound predicate
- When the domain is understood, we don't need to keep repeating it
  - We write

$$\exists a: Q(a)$$

  - or

$$\exists a: (G(a) \land H(a))$$

Negative Quantifiers

- IT IS NOT THE CASE THAT (There EXISTS a creature with blue eyes and blonde hair)
- Same as: “All creatures don’t have blue eyes and blonde hair”

$$\neg(\exists a \in A: Q(a)) \equiv \forall a \in A: \neg Q(a)$$

(≡ means they are equivalent/same)

- IT IS NOT THE CASE THAT (All cars have four wheels)
- Same as: “There is a car which does not have four wheels”

$$\neg\left(\forall c \in C: P(c)\right) \equiv \exists c \in C: \neg P(c)$$

- When you take the negation inside the quantifier and negate the predicate, you must switch quantifiers:

$$\exists \rightarrow \forall$$
$$\forall \rightarrow \exists$$

There is a soulmate for EVERY person

- Define domains and a predicate

$$A = \{a | a\ is\ a\ person\}$$

- $P(a, b)$ = "Person a has as a soul mate person b"
- There is some special person b who is a soul mate to every person a

$$\exists b: (\forall a: P(a, b))$$

- For every person a, they have their own personal soul mate b

$$\forall a: (\exists b: P(a, b))$$

- When quantifiers are mixed, the order in which they appear is important for the meaning
  - Order generally **cannot** be switched

Proofs with quantifiers

- **Claim 1.** $\forall n > 2$: if n is even, then n is a sum of two primes. *(Goldbach, 1742)*
- **Claim 2.** $\exists (a, b, c) \in \mathbb{N}^3 \colon a^2 + b^2 = c^2$
  – where $(a, b, c) \in \mathbb{N}^3$ means triples of natural numbers
- **Claim 3.** $\neg\exists (a, b, c) \in \mathbb{N}^3 \colon a^3 + b^3 = c^3$
- **Claim 4.** $\forall (a, b, c) \in \mathbb{N}^3 \colon a^3 + b^3 = c^3$
- Think about what it would take to prove these claims