What is Computing?

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 23

Overview

- Decision problems
- Languages
  - Describing a language
- Complexity of a computing problem

What is a Computing Problem?

- There are many classes of computing problems
  - And many questions one can ask whose answers can be computed
- For example, *decide* YES or NO whether a given integer $n \in \mathbb{N}$ is prime
  - This is an example of a decision problem
- First, list the primes in increasing order (primes are countable)

$$primes = \{2,3,5,7,11,13,17,19,23, \ldots\}$$

- Here's a quick decision algorithm:
  - Given $n \in \mathbb{N}$, walk through $primes$
  1. If you come to n output YES
  2. If you come to a number bigger than n, output NO
- Not the fastest approach to primality testing, but gets to the heart of computing
- To talk about what is computable, we need to come up with the LANGUAGES of computing!

Decision Problems

Rensselaer

- Consider the set of all primary numbers in binary:
  $$\mathcal{L}_{prime} = \{10,11,101,111,1011,1101,10001,10011,10111,11101, \dots\}$$
  – To answer a question like “Is 9 prime?”, we need to look up the binary representation of 9, 1001, and check if $1001 \in \mathcal{L}_{prime}$
- Consider a push-lamp. Every push toggles between on and off

  – Given the number of pushes, decide whether the light is on or off
  – Encode number of pushes by a binary string, e.g. 101 means 5 pushes
  – Assuming lamp starts in OFF state, what number of pushes correspond to ON?
    - 1, 3, 5, 7, …, i.e., all odd numbers
  – What binary strings correspond to all odd numbers?
  – All strings that end in 1 bit:
  $$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \dots\}$$
  – The light is on after 1010 pushes if and only if $1010 \in \mathcal{L}_{push}$

Decision Problems, cont'd

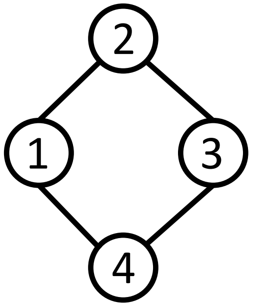
- Consider an electric door with an electric doormat
  - Door opens if you step on the doormat
  - If you step on the doormat, we encode the event as 1
  - If you step off the doormat, we encode the event as 0
  - E.g., 10110 means on, off, on, on, off → open
    - (Two people can step on it at the same time)
  - What are all strings that should lead to the door being open?
  - All strings that start with 1 and have more 1's than 0's

$$\mathcal{L}_{door} = \{1,11,101,110,111,1011,1101,1110,1111, \dots\}$$

  - Given input w, e.g., $w = 1011$, the door is open if and only if $w \in \mathcal{L}_{door}$
- **Decision problems can be formulated as testing membership in a set of strings**

A Decision Problem on Graphs

- (a) **[Optimization]** What's distance between nodes 1 and 3?
  – Answer: 2
- (b) **[Decision]** Is there a path between 1 and 3 of length at most 3?
  – Answer: yes
- Which problem is harder?
  – (a) is harder than (b): (a)'s answer gives (b)'s answer instantly.
- Let's *encode* (b) as a string identifying the graph, nodes of interest and target distance
  – "Is there a path of length at most 3 between nodes 1 and 3 in the graph above"
  – What information does a decision algorithm need?
    - Encode the vertices | edges | start/end nodes | path length



" 1, 2, 3, 4 | (1, 2)(2, 3)(3, 4)(4, 1) | 1, 3 | 3 "

- The graph problem can be encoded as a binary string using ASCII

0011000100101100001100100010110000110011001011000011010001111100001010000001100010010110000110010001010010010100000110010001011000011001 1
00101001001010000001100110010110000110100001010010010100000110100001011000011000100101001011111000011000100101100001100110111111 0000110011

$$\mathcal{L}_{path} = \{All\ strings\ of\ form\ "nodes|edges|endpoints\ of\ path|target\ distance"\ for\ which\\ the\ distance\ between\ the\ endpoints\ in\ the\ graph\ is\ at\ most\ the\ target\ difference\}$$

Is Optimization Really Harder than Decision?

- Can I use the decision problem to obtain an answer for the optimization problem?
- Suppose I ask the decision procedure the following questions
  - Is there a path in the graph between nodes x and y of length at most **1**?
    - Suppose I get back NO
  - Is there a path in the graph between nodes x and y of length at most **2**?
    - Suppose I get back NO
  - Is there a path in the graph between nodes x and y of length at most **3**?
    - Suppose I get back NO
  - Is there a path in the graph between nodes x and y of length at most **4**?
    - Suppose I get back YES
- Keep asking the decision question until the answer is YES
  - The minimum-pathlength between x and y is 4
  - It can take long, but it works.
- Decision and optimization are "equivalent" when it comes to *solvability*.
- **A computing problem is a decision problem.**

Languages

- Standard formulation of a decision problem:
  - **Problem:** GRAPH-DISTANCE-D
  - **Input:** Finite graph G; nodes $x,\ y$; target distance D
  - **Question:** Is there an (x, y)-path in G of length at most D
- Every decision problem has a YES-set, which we usually don't explicitly list

$$\begin{aligned}\text{YES-set} &= \{\text{input strings } w \text{ for which the answer is yes}\}\\ &= \{w_1, w_2, w_3, \dots\}\end{aligned}$$

- A *language* is any set of finite binary strings
- A *computing problem* is a YES-set, a set of *finite* binary strings.

Computing Problems Are Languages

- **Language:** Set of finite binary strings.
- **Solving the problem.** Give a "procedure" to tell if a general input w is in the language (YES-set).
- Abstract, precise and general formulation of a computing problem.
- Examples:
  - A finite language
    $$\{\varepsilon, 1,10,01\}$$
  - All finite strings
    $$\Sigma^* = \{\varepsilon, 0,1,00,01,10,11,000,001,010,011, \ldots\}$$
  - All prime numbers
    $$\mathcal{L}_{prime} = \{10, 11, 101, 111, 1011, 1101, 10001\}$$
  - Push-lamp language
    $$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \ldots\}$$
  - Doormat language
    $$\mathcal{L}_{door} = \{1,11,101,110,111,1011,1101,1110,1111, \ldots\}$$

Computing Problems Are Languages, cont'd

- More examples
  - All unary strings (strings of all 1's)
    $$\mathcal{L}_{unary} = \{1,11,111,1111,\ldots\} = \left\{1^{\bullet n} \middle| n \geq 0\right\}$$
  - All strings with repeated 01 substrings
    $$\mathcal{L}_{(01)^n} = \{01,0101,010101,\ldots\} = \left\{(01)^{\bullet n} \middle| n \geq 0\right\}$$
  - All strings where n 0's are followed by n 1s
    $$\mathcal{L}_{0^n 1^n} = \{01,0011,000111,\ldots\} = \left\{0^{\bullet n} 1^{\bullet n} \middle| n \geq 0\right\}$$
  - All palindromes
    $$\mathcal{L}_{pal} = \{\varepsilon, 0,1,00,11,000,010,101,111,\ldots\}$$

Describing a Language: String Patterns and Variables

- Some languages are easier to describe than others
$$\mathcal{L} = \{01, 0101, 010101, \dots\}$$
- In this case, we can use a variable to formally define $\mathcal{L}$:
$$\mathcal{L} = \left\{w \middle| w = (01)^{\bullet n}, where\ \ n \geq 0\right\}$$
  - (informally, $\left\{(01)^{\bullet n} \middle| n \geq 0\right\}$)
- Some cases are slightly harder (maybe use 2 variables):
$$\left\{u \bullet v \middle| u \in \Sigma^* \ and\ v = u^R\right\} =$$
$$= \{\varepsilon, 00, 11, 0000, 1111, \dots\}$$
  - What is this set?
  - All even palindromes
- **Exercise**. Define
$$\mathcal{L}_{add} = \{0100, 011000, 001000, 00110000, 00010000, 0001100000, 01110000, 0011100000, 000111000000, \dots\}$$
  - Answer: $\{0^{\bullet n} \bullet 1^{\bullet m} \bullet 0^{\bullet n+m}\}$

Regular Expressions

- For more complicated patterns, we use regular expressions
  - e.g. the Unix/Linux command:

    ```
    ls FOCS*
    ```

  - Does anyone know what that command does?
    - lists everything in the folder that starts with `FOCS`
    - the * is a “wild-card”, means “everything”

The Regular Expression: $\{\mathbf{1,11}\}\bullet\overline{\{\mathbf{0,01}\}^*}\bullet(\{\mathbf{00}\} \cup \{\mathbf{1}\}^*)$

- Regular expression basic building blocks are finite languages:

$$\{1{,}11\} \quad \{0{,}01\} \quad \{00\} \quad \{1\}$$

- Combine these using
  - union, intersection, complement
    - So far so good
  - concatenation ($\bullet$), Kleene-star ($*$)
    - Um, OK?
- **Concatenation of languages.**

$$\mathcal{L}_1\bullet\mathcal{L}_2\bullet\mathcal{L}_3 = \{w_1w_2w_3 | w_1 \in \mathcal{L}_1, w_2 \in \mathcal{L}_2, w_3 \in \mathcal{L}_3\}$$

- Example:

$$\{0{,}01\}\bullet\{0{,}11\} = \{00{,}011{,}010{,}0111\}$$

- What about $\{0{,}11\}\bullet\{0{,}01\}$?

$$\{0{,}11\}\bullet\{0{,}01\} = \{00{,}001{,}110{,}1101\}$$

  - Concatenation is not commutative! ($\mathcal{L}_1\bullet\mathcal{L}_2 \neq \mathcal{L}_2\bullet\mathcal{L}_1$)
- Self-concatenation:

$$\{0{,}01\}\bullet\{0{,}01\} = \{0{,}01\}^{\bullet 2} = \{00{,}001{,}010{,}0101\}$$

The Regular Expression: $\{\mathbf{1},\mathbf{11}\}\bullet\overline{\{\mathbf{0{,}01}\}*}\bullet(\{\mathbf{00}\}\cup\{\mathbf{1}\}^{*})$

- Kleene star: All possible concatenations of a finite number of strings from a language

$$\{0{,}01\}^* = \{\varepsilon, 0{,}01{,}00{,}001{,}010{,}0101{,}000, \dots\}$$
$$= \bigcup_{n=0}^{\infty} \{0{,}01\}^{\bullet n}$$

- Similarly,

$$\{1\}^* = \{\varepsilon, 1{,}11{,}111{,}1111{,}11111, \dots\}$$
$$= \bigcup_{n=0}^{\infty} \{1\}^{\bullet n}$$

- To generate 1110111 (from the regular expression in the title):

$$11 \in \{1{,}11\}$$
$$10 \in \overline{\{0{,}01\}^*}$$
$$111 \in \{00\} \cup \{1\}^*$$

- Hence,

$$1110111 \in \{1{,}11\}\bullet\overline{\{0{,}01\}*}\bullet(\{00\} \cup \{1\}^*)$$

Questions Regarding Regular Expressions

- Is there a simple procedure to test if a given string satisfies a regular expression?

$$1110111 \in \{1,11\} \bullet \overline{\{0,01\}*} \bullet (\{00\} \cup \{1\}^*)$$

  - The approach we've seen so far requires a lot of search

- Can we write a regular expression for all palindromes (strings which equal their reversal)?
  - "Reverse" is not an operator in regular expressions

Recursively Defined Languages: Palindromes

- Let's define the palindromes language:

$$\varepsilon, 0,1 \in \mathcal{L}_{palindrome} \qquad \textbf{[basis]}$$

$$w \in \mathcal{L}_{palindrome} \rightarrow 0 \bullet w \bullet 0 \in \mathcal{L}_{palindrome} \qquad \textbf{[constructor rules]}$$

$$\rightarrow 1 \bullet w \bullet 1 \in \mathcal{L}_{palindrome}$$

Nothing else is in $\mathcal{L}_{palindrome}$ **[minimality]**

Recursively Defined Languages, cont'd

- Here are two very similar looking languages:

$$\{0^{\bullet n}, 1^{\bullet k} | n, k \geq 0\}$$
$$\{0^{\bullet n}, 1^{\bullet n} | n \geq 0\}$$

  - Are they the same?
  - Second one only has strings with the same number of 0s and 1s
- These computing problems look similar.
- They are **VERY** different. Which do you think is more "complex"?
  - Turns out the second one is much harder. Why?
  - We need to remember the number of 0s in order to check if the number of 1s is the same
- How to define complexity of a computing problem?

Complexity of a Computing Problem

- Remember the lamp-push language:

$$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \dots\}$$

  - (strings ending in 1)
- We define a difficult set intuitively as a set where:
  - We have a “complex” YES-set
  - It is computationally “hard” to test membership in YES-set
- How do we test membership? That brings us to *Models Of Computing*.
- Suppose we are given a string 1101
  - Is it in the push language?
  - Yes
  - Turns out we can come up with a very simple device to test membership

Complexity of a Computing Problem, cont'd

- Remember the lamp-push language:
  $$\mathcal{L}_{push} = \{1{,}01{,}11{,}001{,}011{,}101{,}111{,}0001{,}0011{,}0101{,}0111{,}1001, \ldots\}$$
  - (strings ending in 1)
- Suppose we are given a string 1101
- Consider the following machine with two states

- Depending on the machine's state, there are four possible rules:

Complexity of a Computing Problem, cont'd

- Remember the lamp-push language:

$$\mathcal{L}_{push} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, \dots\}$$

  – (strings ending in 1)
- Suppose we are given a string 1101
- Consider the following machine with two states

- Depending on the machine's state, there are four possible rules:
  1. In state q_0, when you process a 0, transition to state q_0

Complexity of a Computing Problem, cont'd

- Remember the lamp-push language:
  $$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \dots\}$$
  - (strings ending in 1)
- Suppose we are given a string 1101
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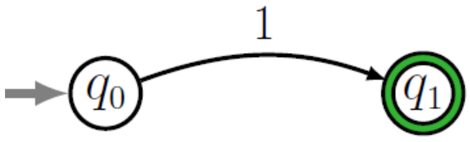

- Depending on the machine's state, there are four possible rules:
  1. In state q_0, when you process a 0, transition to state q_0
  2. In state q_0, when you process a 1, transition to state q_1

Complexity of a Computing Problem, cont'd

- Remember the lamp-push language:

  $$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \ldots\}$$

  – (strings ending in 1)
- Suppose we are given a string 1101
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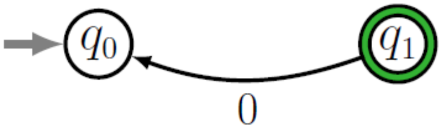

- Depending on the machine's state, there are four possible rules:
  1. In state q_0, when you process a 0, transition to state q_0
  2. In state q_0, when you process a 1, transition to state q_1
  3. In state q_1, when you process a 0, transition to state q_0

Complexity of a Computing Problem, cont'd

- Remember the lamp-push language:
  $$\mathcal{L}_{push} = \{1,01,11,001,011,101,111,0001,0011,0101,0111,1001, \ldots\}$$
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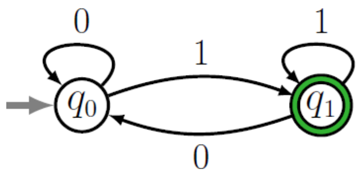
- Suppose we are given a string 1101
- Consider the following machine with two states

- Depending on the machine's state, there are four possible rules:
  1. In state q_0, when you process a 0, transition to state q_0
  2. In state q_0, when you process a 1, transition to state q_1
  3. In state q_1, when you process a 0, transition to state q_0
  4. In state q_1, when you process a 1, transition to state q_1
- Full machine is then

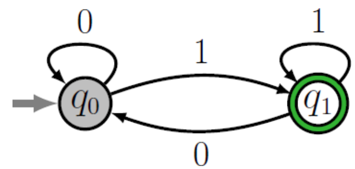


A Simple Computing Machine (DFA)

- DFA stands for deterministic finite automaton
- Let's see how the computing machine works
- We process the string in order
  - After each symbol we perform a transition in the DFA
  - We start from the DFA initial state
  - If we end in an accepting state, the string is accepted, i.e., it is in the language

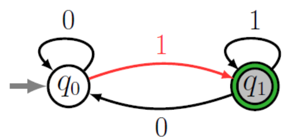
A Simple Computing Machine, cont'd

- Start the machine in the initial state



- Then process the string in order

1101



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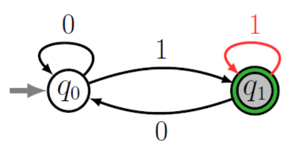

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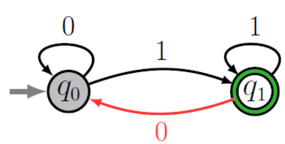

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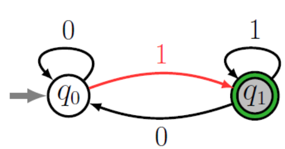
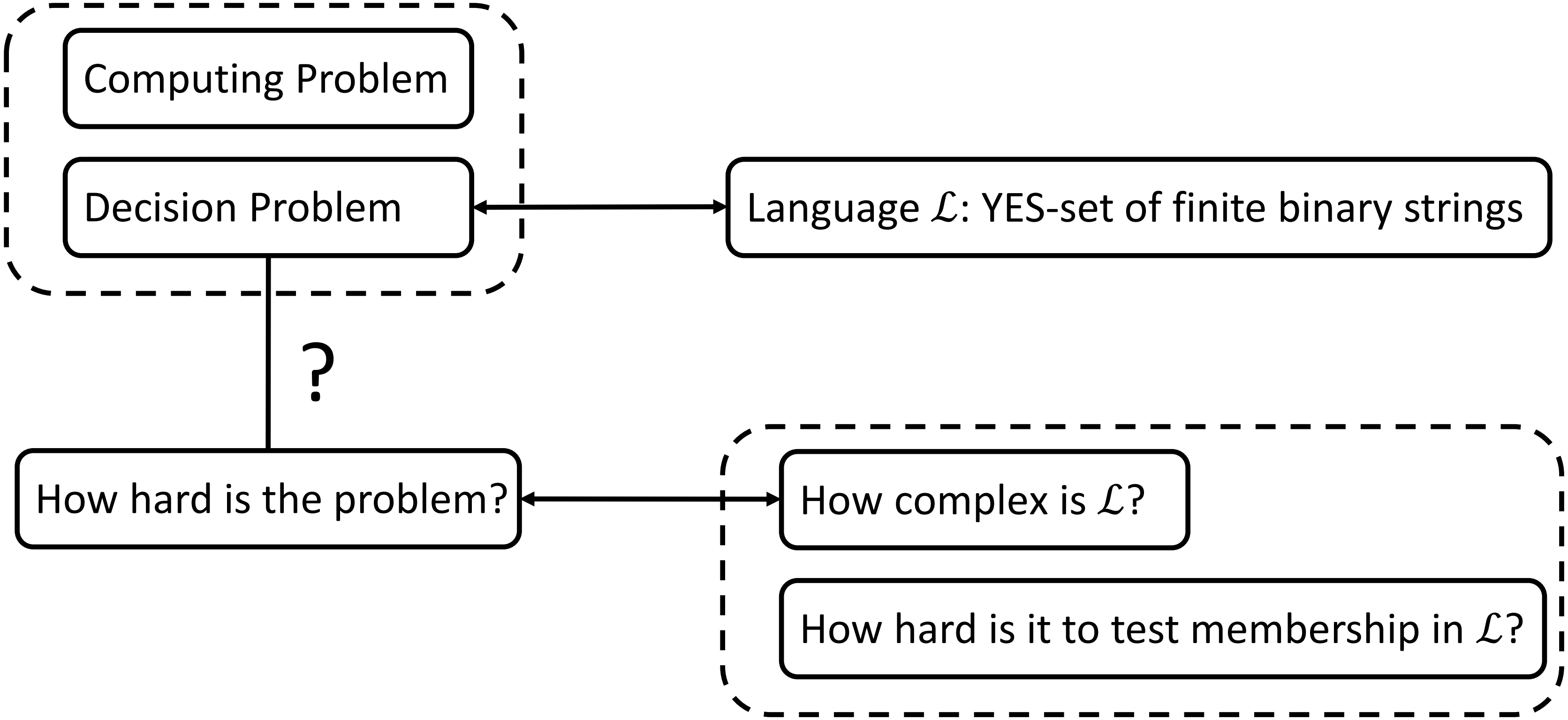


- Remember the lamp-push language:
$\mathcal{L}_{push} = \{1, 01, 11, 001, 011, 101, 111, 0001, 0011, 0101, 0111, 1001, \ldots\}$
- Strings in $\mathcal{L}_{push}$ end in "accepting" state q_1
- Strings not in $\mathcal{L}_{push}$ do not

Computing Problems and Their Difficulty



Computing Problems and Their Difficulty, cont'd

- A problem can be harder in two ways.
  1. The problem needs more resources.
     - For example, the problem can be solved with a similar machine to ours, except with more states.
  2. The problem needs a different *kind* of computing machine, with superior capabilities.
- The first type of "harder" is the focus of a follow-on algorithms course.
- We focus on what *can and can't be solved* on a particular kind of machine.