Infinity

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 22

Summary of Our Stroll Through Discrete Math

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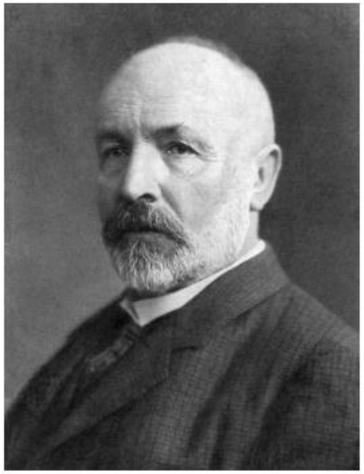
- Precise statements, proofs and logic.
- **INDUCTION**.
- Recursively defined structures and Induction. (Data structures; PL)
- Sums and asymptotics. (Algorithm analysis)
- Number theory. (Cryptography; probability; fun)
- Graphs. (Relationships/conflicts; resource allocation; routing; scheduling,. . .)
- Counting. (Enumeration and brute force algorithms)
- Probability. (Real world algorithms involve randomness/uncertainty)
  - Inputs arrive in a random order;
  - Randomized algorithms (primality testing, machine learning, routing, conflict resolution . . .)
  - Expected value is a summary of what happens. Variance tells you how good the summary is.

Today: Infinity

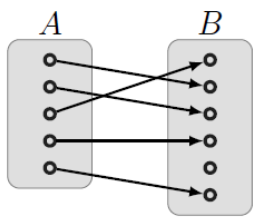
- Comparing "sizes" of sets: countable.
  - Rationals are countable.
- Uncountable
  - Infinite binary strings.
- What does Infinity have to do with computing?



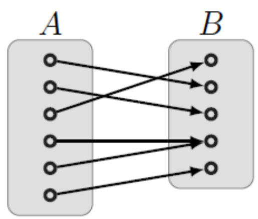
Georg Cantor

“Size” of a Set: Cardinality

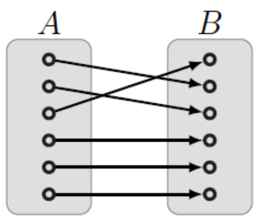
- There’s a reason why small kids use fingers to count
  - They map their intuitive knowledge of 2-fingers to 2 of another object
- You have an *equal* number of fingers on each hand
  - You can map your left hand’s fingers to your right hand’s fingers
- Recall some types of such maps



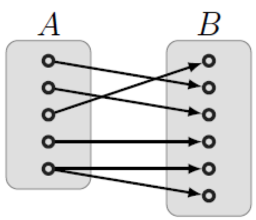
1-1, but **not** onto.
(injection, $A \xrightarrow{INJ} B$)
$|A| \le |B|$



onto; **not** 1-1
(surjection, $A \xrightarrow{SUR} B$)
$|A| \ge |B|$

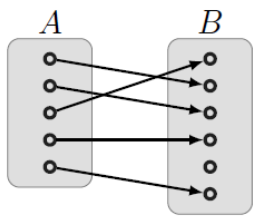


onto **and** 1-1
(bijection, $A \xrightarrow{BIJ} B$)
$|A| = |B|$



not a function

"Size" of a Set: Cardinality



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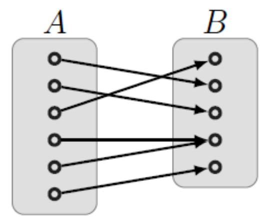

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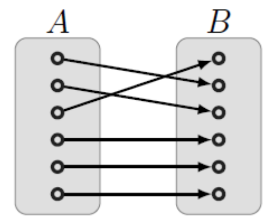

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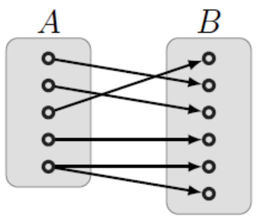


not a function

- *Cardinality* $|A|$ ("size"), read "cardinality of *A*," is the number of elements for finite sets
- In general, we can define the following relations between sets:

$|A| \leq |B|$ iff there is an injection (1-to-1) from A to B, i.e., $f: A \xrightarrow{INJ} B$

$|A| > |B|$ iff there is no injection (1-to-1) from A to B

$|A| \geq |B|$ iff there is a surjection (onto) from A to B, i.e., $f: A \xrightarrow{SUR} B$

$|A| = |B|$ iff there is a bijection (1-1 and onto) from A to B, i.e., $f: A \xrightarrow{BIJ} B$

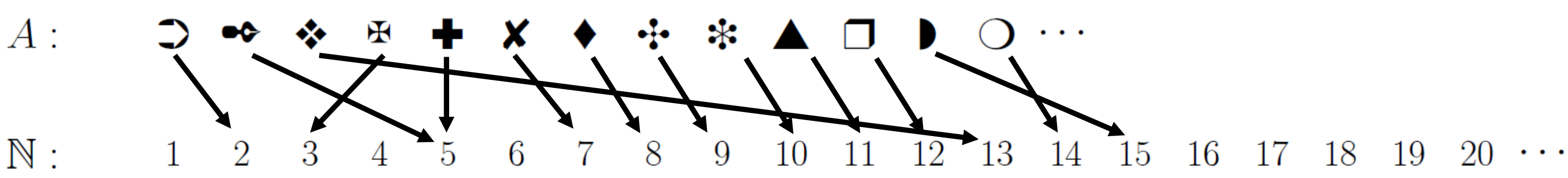
$|A| \leq |B|\ and\ |B| \leq |A| \rightarrow |A| = |B|$ [**Cantor-Bernstein Theorem**]

A Countable Set's Cardinality is at most $|\mathbb{N}|$

- Suppose we have a finite set $A = \{a_1, \dots, a_n\}$
  - Cardinality is $|A| = n$ if and only if there is a bijection from A to $\{1, \dots, n\}$
  - Can you come up with such a function?

$$f(a_i) = i$$

- For infinite sets: the set A is countable if $|A| \leq |\mathbb{N}|$.
  - Intuitively, A is "smaller than" $\mathbb{N}$
  - Sometimes we say A is at most countable to include both finite and infinite sets that are "smaller than" $\mathbb{N}$
- To show that A is countable you must find a 1-to-1 mapping from A to $\mathbb{N}$



- You cannot skip over any elements of A, but you might not use every element of $\mathbb{N}$

A Countable Set's Cardinality is at most $|\mathbb{N}|$, cont'd

- To prove that a function $f: A \mapsto \mathbb{N}$ is an injection:
  - Assume f is *not* an injection. (Proof by contradiction.)
  - This means there is a pair $x, y \in A$ for which $x \neq y$ and $f(x) = f(y)$
  - Use $f(x) = f(y)$ to prove that $x = y$, a contradiction. Hence, f *is* an injection

All Finite Sets are Countable

- Suppose $A = \{3,6,8\}$
  - To show $|A| \leq |\mathbb{N}|$, we give an injection from A to $\mathbb{N}$

$$3 \mapsto 1, 6 \mapsto 23134, 8 \mapsto 8$$

- For an arbitrary set $A = \{a_1, \ldots, a_n\}$

$$a_1 \mapsto 1, a_2 \mapsto 2, \ldots, a_n \mapsto n$$

Non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ are countable

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- How can this be??
  - I know for a fact that $\mathbb{N}_0$ contains every element in $\mathbb{N}$, plus an extra 0
  - It's clearly bigger!!
- Well, if they were finite sets, I would agree. But let's recall the definition.
- To prove that $|\mathbb{N}_0| \leq |\mathbb{N}|$, we need an injection $f: \mathbb{N}_0 \overset{INJ}{\longmapsto} \mathbb{N}$
  - Ideas?
  - Let's try $f(x) = x + 1, \forall x \in \mathbb{N}_0$
- *Proof*.
  - Assume f is not an injection. So, there are $x \neq y$ in $\mathbb{N}_0$ with $f(x) = f(y)$:

$$x + 1 = f(x) = f(y) = y + 1$$

  - But that means $x + 1 = y + 1$, i.e., $x = y$. Contradiction.
- Also, we know that $|\mathbb{N}| \leq |\mathbb{N}_0|$ since $\mathbb{N} \subseteq \mathbb{N}_0$
- By the Cantor-Bernstein Theorem, $\mathbb{N} = \mathbb{N}_0$

$\mathbb{N}_0$:	0	1	2	3	4	5	6	7	8	9	$\cdots$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	$\cdots$

Positive Even Numbers and Integers and Countable

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- Huh? The even numbers are exactly half of all natural numbers!!

$$E = \{2,4,6,\dots\}, \text{ so surely } |E| = \frac{1}{2}\mathbb{N}!!$$

- Turns out, not quite. Can you see a bijection?
  - The bijection $f(x) = \frac{1}{2}x$ proves $|E| = |\mathbb{N}|$

E :	2	4	6	8	10	12	14	16	18	20	⋯
	↕	↕	↕	↕	↕	↕	↕	↕	↕	↕	
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	⋯

- OK, fine, but all integers?? It has to be the case that $|\mathbb{Z}| = 2|\mathbb{N}|$!!
  - Is there a bijection? ☺
  - Recall the integers are $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$

	0		-1		-2		-3		-4		-5		-6		-7		-8		-9		-10 ⋯
	↕		↕		↕		↕		↕		↕		↕		↕		↕		↕		↕
$\mathbb{N}$:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21 ⋯
		↕		↕		↕		↕		↕		↕		↕		↕		↕		↕	
		1		2		3		4		5		6		7		8		9		10 ⋯	

- **Exercise.** What is a mathematical formula for the bijection?

Every Countable Set Can Be "Listed"

- What does it mean for a set to be "listed"?
  - You know the exact position of each element in the set
  - Regardless of whether the set is finite or infinite
- For example:
  - $\{3,6,8\}$ is a list (why?)
    - (because it's a finite set)
  - $E = \{2,4,6, \ldots\}$ is a list (why?)
    - element i is just $2i$
  - What about $\mathbb{Z}$?
    - Suppose I represent it as $\{\ldots, -3, -2, -1,0,1,2,3, \ldots\}$
      - Unclear what the indices are
    - Suppose I represent it as $\{0, -1,1, -2,2, \ldots\}$
      - Element 1 is 0, o.w. it is $-i/2$ (if is even), $(i-1)/2$ (if i is odd)

Every Countable Set Can Be "Listed", cont'd

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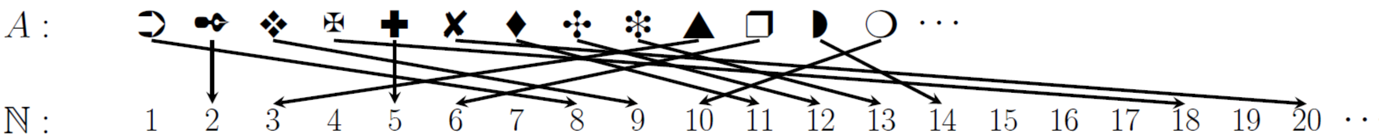
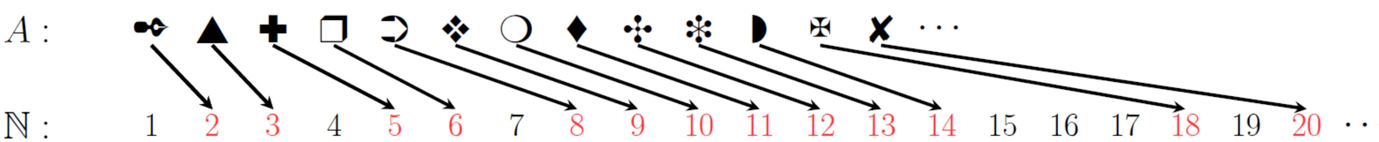
- Suppose I give you the following mapping between sets A and $\mathbb{N}$



  - How do I "list" A?
  - Order elements according to their assigned value



- In general, a set can be "listed" if
  - Different elements are assigned to different list-positions.
  - We can determine the list-position of *any* element in the set.

Union of Two Countable Sets is Countable

- Consider two countable sets, A and B
  - They are countable, so I can write $A = \{a_1, a_2, a_3, \dots\}$ and $B = \{b_1, b_2, b_3, \dots\}$
- Now, let's look at the union

$$A \cup B = \{a_1, a_2, a_3, \dots, b_1, b_2, b_3, \dots\}$$

  - Hm, how do I show this is countable?
  - Can't use "…" twice
  - How do I reorder terms?
  - Need to know the position of each b_i
- Here's a better reordering:

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$$

  - Now I know the position of each element
  - List-position of a_i is $2i - 1$
  - List-position of b_i is $2i$
- **Exercise.** Get a list of $\mathbb{Z}$ with $A = \{0, -1, -2, -3, \dots\}$ and $B = \{1, 2, 3, \dots\}$ using union.

Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$

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- OK, this one is very surprising!
  - There are **infinitely** many rationals between every two integers!
  - The rationals are dense (there is a rational between any two rationals)!
  - Natural numbers are not!
  - Also, the set of rationals can be expressed as the product of integers and natural numbers:
    - $|\mathbb{Q}| = |\mathbb{N}| \times |\mathbb{Z}|$, so of course $|\mathbb{Q}| \gg |\mathbb{N}|$??
- Well, let's see…

Rationals are Countable: $|\mathbb{Q}| = |\mathbb{N}|$, cont'd

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$\mathbb{Z}$

$\mathbb{N}$

$\mathbb{Q}$	0	1	-1	2	-2	3	-3	4	-4
1	$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$
2	$\frac{0}{2}$	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{2}$	$\frac{-2}{2}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{2}$	$\frac{-4}{2}$
3	$\frac{0}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{3}$	$\frac{-3}{3}$	$\frac{4}{3}$	$\frac{-4}{3}$
4	$\frac{0}{4}$	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{4}$	$\frac{-2}{4}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{4}$	$\frac{-4}{4}$
5	$\frac{0}{5}$	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{5}$	$\frac{-4}{5}$
…	…	…	…	…	…	…	…	…	…

- How do I "list" all rationals?
  - I need a list that visits each rational *exactly* once!
  - I need to know each rational's position *exactly*

$$\mathbb{Q} = \left\{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{0}{2}, \frac{0}{3}, \frac{1}{3}, \dots\right\}$$

$$|\{Rational\ Values\}| \leq |\mathbb{Q}| \leq |\mathbb{N}|$$

- **Exercise.** What is a mathematical formula for the list-position of $z/n \in \mathbb{Q}$?

Programs are Countable

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- Programs are finite binary strings. We show that all finite binary strings $\mathcal{B}$ are countable
  - How do I list them?
    - Start with the empty string (duh…)
    - Then list all strings of length 1, length 2, etc.

    $\mathcal{B} = \{\varepsilon, 0,1,00,01,10,11,000,001,010,011,100,101,110,111, \ldots\}$
- I now know the exact position of every string!
- **Exercise.** What is the list-position of 0110?
- **Exercise**. For the $(k+1)$-bit string $b = b_k b_{k-1} \cdots b_1 b_0$, define the string's numerical value:

$$value(b) = b_0 \cdot 2^0 + b_1 \cdot 2^1 + \cdots + b_{k-1} 2^{k-1} + b_k 2^k$$

  - Show:

$$\text{list-position of } b = 2^{length(b)} + value(b)$$

- Wait a second… We keep seeing larger and larger sets that are countable!!

$$\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathcal{B}$$

  - SURELY EVERYTHING IS COUNTABLE!?!

Infinite Binary Strings Are Uncountable!

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- One of the most cool results in computational theory
- Cantor's Diagonal Argument: Assume there is a list of *all* infinite binary strings

$$
\begin{array}{lccccccccccccccc}
b_1: & \mathbf{0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
b_2: & 0 & \mathbf{0} & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\cdots \\
b_3: & 1 & 1 & \mathbf{0} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0\cdots \\
b_4: & 1 & 0 & 1 & \mathbf{0} & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\cdots \\
b_5: & 0 & 1 & 1 & 0 & \mathbf{1} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\cdots \\
b_6: & 0 & 1 & 0 & 1 & 1 & \mathbf{0} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\cdots \\
b_7: & 0 & 0 & 1 & 0 & 0 & 1 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\cdots \\
b_8: & 0 & 0 & 1 & 0 & 1 & 1 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 1 & 0 & 0\cdots \\
b_9: & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & \mathbf{0} & 0 & 1 & 0 & 0 & 0 & 0\cdots \\
b_{10}: & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & \mathbf{1} & 1 & 0 & 0 & 0 & 0\cdots \\
\vdots & & & & & & & & & & & & & & &
\end{array}
$$

- We'll now show that there exists a string that cannot be in that list!
- Look at the (red) diagonal string:

$$b = 0000100101\cdots$$

- What's so special about this string?
  - Let's flip the bits

$$\bar{b} = 1111011010\cdots$$

  - This string is not in the list!
  - Differs from each string b_i in position i

The Real Numbers are Uncountable

- Every real between 0 and 1 has an infinite binary representation and every infinite binary string evaluates to a real number

$$\text{e.g., } 0.0011111111111111\cdots = \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \cdots = \frac{1}{4}$$

- This means

$$|\{reals\ in\ [0,1]\}| = |\{infinite\ binary\ strings\}| > |\mathbb{N}|$$

- Aha, found one!
- **Brain-breaking exercise (Continuum Hypothesis)**. Prove that there is no set $\mathcal{R}$ s.t.

$$|\{reals\ in\ [0,1]\}| > |\mathcal{R}| > |\mathbb{N}|$$

Infinity and Computing

- Cantor took on the abstract beast Infinity. (1874)
- ~60 years later, Alan Turing asked the abstract question: What can we compute? (1936)
- For example, consider the set of binary functions f defined on $\mathbb{N}$

n:	1	2	3	4	5	6	7	8	9	10	$\cdots$
$f(n)$:	0	1	1	0	1	0	0	0	1	1	$\cdots$

  - Turns out the set of all such functions is uncountable
    - Corresponds to the set of all infinite binary strings
- Every program is a finite binary string. For example:

```
int main();     //a program that does nothing
```

  - This program corresponds to the finite binary string (ASCII code)

    01101001011011100111010000100000011011010110000101101001011011100010100000101001001110 11

- So, the number of programs is countable
  - But the number of functions is uncountable!
- There are many more functions than we can write/compute!

Infinity and Computing, cont'd

- There are MANY MANY functions that cannot be computed by programs!
- Are there interesting, useful functions that cannot be computed by programs?