Expected Value of a Sum

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 20

Overview

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- Expected value of a sum
  - Sum of dice
  - Binomial
  - Waiting time
  - Coupon collecting
- Iterated expectation
- Build-up expectation
- Expected value of a product
- Sum of indicators

Expected Value of a Sum

- Suppose you play the lottery. If you get two tickets, you expect to win two times as much money
- If you flip two coins, you expect to get two times as many H's (1 vs. 0.5)
- **The expected value of a sum is a sum of the expected values.**
- *Theorem [Linearity of Expectation]*. Let $X_1, X_2, \ldots, X_k$ be random variables and let $Z = a_1X_1 + a_2X_2 + \cdots + a_kX_k$ be a linear combination of the X_i's. Then,
$$\mathbb{E}[Z] = a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2] + \cdots + a_k\mathbb{E}[X_k]$$
- *Proof*.

$$\begin{aligned}
\mathbb{E}[Z] &= \sum_{\omega\in\Omega} \big(a_1X_1(\omega) + a_2X_2(\omega) + \cdots + a_kX_k(\omega)\big) \cdot \mathbb{P}[\omega] \\
&= a_1 \sum_{\omega\in\Omega} X_1(\omega) \cdot \mathbb{P}[\omega] + a_2 \sum_{\omega\in\Omega} X_2(\omega) \cdot \mathbb{P}[\omega] + \cdots + a_k \sum_{\omega\in\Omega} X_k(\omega) \cdot \mathbb{P}[\omega] \\
&= a_1\mathbb{E}[X_1] + a_2\mathbb{E}[X_2] + \cdots + a_k\mathbb{E}[X_k]
\end{aligned}$$

- QED.

Expected Value of a Sum, cont'd

- Summation can be taken inside or pulled outside an expectation
- Constants can be taken inside or pulled outside an expectation

$$\mathbb{E}\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i \mathbb{E}[X_i]$$

Sum of Dice

- Let X be the sum of 4 fair dice. What is $\mathbb{E}[X]$?
- Let's list the outcome tree!

sum	4	5	6	7	…	24
$\mathbb{P}[sum]$	$\frac{1}{1296}$	$\frac{4}{1296}$	$\frac{10}{1296}$	?	…	$\frac{1}{1296}$

- So the expected value is,

$$\mathbb{E}[X] = 4 \times \frac{1}{1296} + 5 \times \frac{4}{1296} + \cdots$$

  – Um, this is going to take a while…

- MUCH faster to observe that X is a sum:

$$X = X_1 + X_2 + X_3 + X_4$$

  – where X_i is the value rolled by die i
  – what is the expected value of each X_i?

$$\mathbb{E}[X_i] = 3.5$$

- By Linearity of Expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = 14$$

in general $n \times 3.5$

Sum of Dice Exercise

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- **Exercise.** Compute the full PDF for the sum of 4 dice and expected value from the PDF.

Expected Number of Successes in n Coin Tosses

- Suppose we have n trials, and the probability of success is p
- Let X denote the number of successes

$$X = X_1 + X_2 + \cdots + X_n$$

- Each X_i is a Bernoulli and

$$\mathbb{E}[X_i] = p$$

  – Why?

$$\mathbb{E}[X_i] = 1 \times p + 0 \times (1 - p)$$

- By linearity of expectation,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= n \times p \end{aligned}$$

Expected Waiting Time to n Successes

- Let X be the waiting time for n successes with success probability p
- How do we define X formally?

$$X = X_1 + X_2 + \cdots + X_n$$

  – where X_1 is the waiting time until the first success
  – where X_2 is the waiting time from the first to the second success
- Each X_i is a waiting time to *one* success
  – What is $\mathbb{E}[X_i]$?

$$\mathbb{E}[X_i] = \frac{1}{p}$$

- By linearity of expectation:

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{E}[X_1 + X_2 + \cdots + X_n] \\ &= \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n] \\ &= \frac{n}{p}\end{aligned}$$

- **Example.** If you are waiting for 3 heads, you have to wait 3-times as long as for 1 head.
- **Exercise.** Compute the expected *square* of the waiting time.

Coupon Collection

- When I was a kid, when the Soccer World Cup started, we would try to collect stickers of all the players
  - Each pack had 5 stickers, but there were loooots of repeats
- Kids also would collect country flags from gum packs
  - There were 169 flags in total, but lots of repeats again
  - How many gum-purchases would one have to make on average?
- Let X be the waiting time to collect all 169 flags

$$X = X_1 + X_2 + \cdots + X_n$$

  - where X_1 is the waiting time until the first success
  - where X_2 is the waiting time from the first to the second success, etc.
- Are probabilities of success the same?
  - Chance of repeats increases after each new flag

Coupon Collection, cont'd

- Let X be the waiting time to collect all 169 flags

$$X = X_1 + X_2 + \cdots + X_n$$

  - where X_1 is the waiting time until the first success
  - where X_2 is the waiting time from the first to the second success, etc.
- What is $\mathbb{E}[X_1]$?

$$\mathbb{E}[X_1] = 1 = \frac{n}{n}$$

  - No repeats
- The probability of the 2nd success is $\frac{n-1}{n}$, so $\mathbb{E}[X_2] = \frac{n}{n-1}$
- The probability of the 3rd success is $\frac{n-2}{n}$, so $\mathbb{E}[X_3] = \frac{n}{n-2}$
- By linearity of expectation:

$$\mathbb{E}[X] = n \times \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{1}\right) = nH_n \approx n(\ln n + 0.577)$$

  - When $n = 169$, you expect to buy 965 gum packs! Better have strong teeth!

Coupon Collection Example

- **Example.** Cereal box contains 1-of-5 cartoon characters. Collect all to get $2 rebate
  - Expect to buy about 12 cereal boxes
  - If a cereal box costs $5, that's a whopping 3.3% discount

Iterated Expectation

- Here's a convoluted experiment (probabilists like twisted expectations…)
- **Experiment.** Roll a die and let X_1 be the value. Now, roll a second die X_1 times and let X_2 be the sum of these X_1 rolls of the second die.
  - An example outcome is $(4; 2{,}1{,}2{,}6)$
    - with $X_1 = 4, X_2 = 11$
- What is $\mathbb{E}[X_2]$?
  - Can use the law of total expectation to calculate it. How?
$$\mathbb{E}[X_2] = \mathbb{E}[X_2|X_1 = 1] \times \mathbb{P}[X_1 = 1] + \cdots + \mathbb{E}[X_2|X_1 = 6] \times \mathbb{P}[X_1 = 6]$$
- Let's look at another approach. What is the conditional expectation $\mathbb{E}[X_2|X_1]$?
$$\mathbb{E}[X_2|X_1] = X_1 \times 3.5$$
  - The RHS is a *function* of X_1, a random variable. Compute its expectation.
- Another version of the law of total expectation lets us write
$$\begin{aligned}\mathbb{E}[X_2] &= \mathbb{E}_{X_1}\big[\mathbb{E}[X_2|X_1]\big]\\ &= \mathbb{E}[X_1] \times 3.5\\ &= 3.5 \times 3.5 = 12.25\end{aligned}$$

Build-Up Expectation: Waiting for 2 Hs and 6 Ts

- Probabilists also like waiting!
  - And while they wait, they like to calculate how long they are expected to wait
- Suppose we have a potentially biased coin, $\mathbb{P}[H] = p$
- Let's introduce the relevant notation:

$$W(k, l) = \mathbb{E}[waiting\ time\ to\ k\ Hs\ and\ l\ Ts]$$

- The first toss is either a H or a T, so by total expectation:

$$\begin{aligned} W(k, l) &= \mathbb{E}[waiting\ time|H] \times \mathbb{P}[H] + \mathbb{E}[waiting\ time|T] \times \mathbb{P}[T] \\ &= 1 + pW(k-1, l) + (1-p)W(k, l-1) \end{aligned}$$

- Aha, this is a recursion similar to the candies counting problem!
  - What are the base cases:

$$W(0, l) = \frac{l}{1-p}$$

$$W(k, 0) = \frac{k}{p}$$

Build-Up Expectation: Waiting for 2 Hs and 6 Ts, cont'd

- Suppose we have a potentially biased coin, $\mathbb{P}[H] = p$
- Let's introduce the relevant notation:

$$W(k,l) = \mathbb{E}[\textit{waiting time to } k \textit{ Hs and } l \textit{ Ts}]$$

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$$W(k,0) = \frac{k}{p}$$

l

	$W(k,l)$	0	1	2	3	4	5	6	7	…
	0	0	2	4	6	8	10	12	14	…
k	1	2								
	2	4								
	…	…								

Build-Up Expectation: Waiting for 2 Hs and 6 Ts, cont'd

- Suppose we have a potentially biased coin, $\mathbb{P}[H] = p$
- Let's introduce the relevant notation:

$$W(k,l) = \mathbb{E}[\textit{waiting time to } k \textit{ Hs and } l \textit{ Ts}]$$

- The first toss is either a H or a T, so by total expectation:

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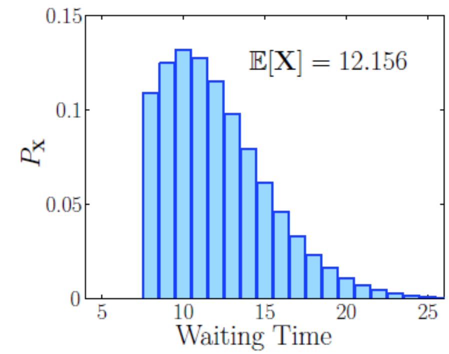

l

$W(k,l)$	0	1	2	3	4	5	6	7	...
0	0	2	4	6	8	10	12	14	...
1	2	3	4.5	6.25	8.13	10.1	12.0	14.0	...
2	4	4.5	5.5	6.88	8.5	10.3	12.2	14.1	...
...	...	...	...	...	...	...	...	...	...

k

$\times p$

$\times (1-p)$

Expected Value of a Product

- Suppose I roll a single die (phew, simple!)
  - Wait for it…
  - What is the expected value of the squared die roll?
    - What are the outcomes?
    - $X = 1 \left(\mathbb{P}[X = 1] = \frac{1}{6}\right), \dots, X = 36 \left(\mathbb{P}[X = 36] = \frac{1}{6}\right)$

$$\mathbb{E}[X^2] = \frac{1}{6} \times 1 + \frac{1}{6} \times 4 + \cdots + \frac{1}{6} \times 36 = 15.16$$

- BEWARE!
  - Expectation is linear, but it's not quadratic (i.e., it does not distribute inside)

$$\mathbb{E}[X] \times \mathbb{E}[X] = 12.25 \neq \mathbb{E}[X^2] = 15.16$$

- Expectation does distribute inside if the variables are independent!
- Let X_1 and X_2 be two independent rolls

$$\begin{aligned}\mathbb{E}[X_1 X_2] &= \frac{1}{36}(1 + 2 + \cdots + 6 + 2 + 4 + \cdots + 8 + \cdots + 6 + 12 + \cdots + 36) \\ &= \frac{441}{36} = 12.25 \\ &= \mathbb{E}[X_1]\mathbb{E}[X_2] = (3.5)^2\end{aligned}$$

Die 2 Value

6	12	18	24	30	36
5	10	15	20	25	30
4	8	12	16	20	24
3	6	9	12	15	18
2	4	6	8	10	12
1	2	3	4	5	6

Die 1 Value

Expected Value of a Product, cont'd

- Expected value of a product XY
  - In general, the expected product is not a product of expectations.
  - For independent random variables, it is: $\mathbb{E}[XY] = \mathbb{E}[X] \times \mathbb{E}[Y]$
  - Why?

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} xy\mathbb{P}[X = x, Y = y] \\
&= \sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} xy\mathbb{P}[X = x]\mathbb{P}[Y = y] \\
&= \sum_{x \in X(\Omega)} x\mathbb{P}[X = x] \sum_{y \in Y(\Omega)} y\mathbb{P}[Y = y] \\
&= \mathbb{E}[X]\mathbb{E}[Y]
\end{aligned}$$

Sum of Indicators: Successes in a Random Assignment

- Another fun experiment!
  - Consider 4 people with hats. I take their hats and throw them randomly. What is the expected number of hats that land on a correct head?
    - (Assuming hats only land on heads!)
- Let X be the number of correct hats when 4 hats randomly land on 4 heads

Hat #

Person # 1 2 3 4

$X_1 = 0$ $X_2 = 1$ $X_3 = 1$ $X_4 = 0$

$$X = X_1 + X_2 + X_3 + X_4$$

- What is the distribution of each X_i?
  - Bernoully, with $\mathbb{P}[X_i = 1] = \frac{1}{4}$
    - Hats are randomly distributed
- By linearity of expectation:

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \mathbb{E}[X_3] + \mathbb{E}[X_4] = 1$$

Sums of Indicators Exercises

- **Exercise.** What about if there are n people?
- **Interesting Example (see text).** Apply sum of indicators to breaking of records
- **Instructive Exercise.** Compute the PDF of X and the expectation from the PDF