Random Variables

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 18

Overview

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- What is a random variable?
- Probability distribution function (PDF) and Cumulative distribution function (CDF).
- Joint probability distribution and independent random variables
- Important random variables
  - Bernoulli: indicator random variables
  - Uniform: simple and powerful. An equalizing force
  - Binomial: sum of independent indicator random variables
  - Exponential: the waiting time to the first success

A Random Variable is a "Measurable Property"

- Temperature: "measurable property" of random positions and velocities of molecules
- Toss 3 coins
  - Define a variable X, to count number of heads (e.g., number-of-heads(HTT) = 1)
  - Define a variable Y, which is 1 if all tosses match (all-tosses-match(HTT) = 0)
  - Define a variable Z, which is doubled for each H and halved for each T

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathbb{P}[\omega]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X(\omega)$	3	2	2	1	2	1	1	0
$Y(\omega)$	1	0	0	0	0	0	0	1
$Z(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$

A Random Variable is a "Measurable Property", cont'd

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathbb{P}[\omega]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X(\omega)$	3	2	2	1	2	1	1	0
$Y(\omega)$	1	0	0	0	0	0	0	1
$Z(\omega)$	8	2	2	$\frac{1}{2}$	2	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{8}$

- Can define events based on these random variables

$$\{X = 2\} = \{HHT, HTH, THH\}$$

$$\mathbb{P}[X = 2] = \frac{3}{8}$$

$$\{X \geq 2\} = \{HHH, HHT, HTH, THH\}$$

$$\mathbb{P}[X \geq 2] = \frac{1}{2}$$

$$\{Y = 1\} = \{HHH, TTT\}, \mathbb{P}[Y = 1] = \frac{1}{4}$$

$$\{X \geq 2\ AND\ Y = 1\} = \{HHH\}, \mathbb{P}[X \geq 2\ AND\ Y = 1] = \frac{1}{8}$$

Probability Distribution Function (PDF)

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- A random variable is a function from the space of outcomes, Ω, to the reals, $\mathbb{R}$
- For example, X maps $\Omega \rightarrow X(\Omega)$

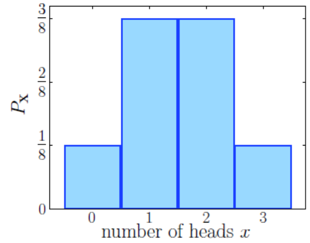
$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \xrightarrow{X} \{3,2,1,0\}$$

- Each *possible* value x of the random variable X corresponds to an event

x	0	1	2	3
Event	$\{TTT\}$	$\{HTT, THT, TTH\}$	$\{HHT, HTH, THH\}$	$\{HHH\}$

- For each $x \in X(\Omega)$, compute $\mathbb{P}[X = x]$ by adding the outcome-probabilities

x	0	1	2	3
$P_X[x]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



Probability Distribution Function (PDF), cont'd

- **Probability Distribution Function (PDF).** The probability distribution function $P_X(x)$ is the probability for the random variable X to take value x,

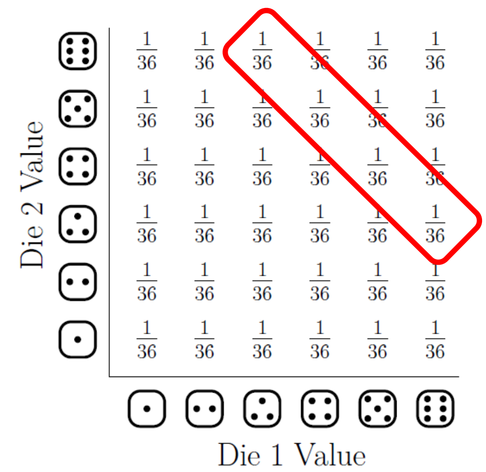
$$P_X(x) = \mathbb{P}[X = x]$$

- Note: typically the abbreviation PDF is used for probability density function
  - i.e., when the variable can take on infinitely many values

PDF for the Sum of Two Dice

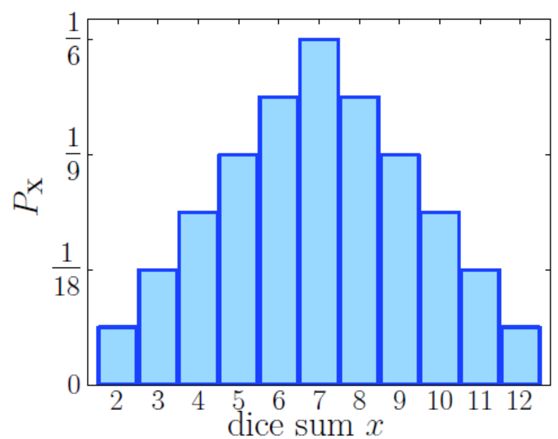
- Define the random variable X which is the sum of two dice
  - How many outcomes does the event $\{X = 9\}$ contain?

$$\mathbb{P}[X = 9] = 4 \times \frac{1}{36} = \frac{1}{9}$$



- What are all possible sums?
  - Possible sums are $X \in \{2, 3, \ldots, 12\}$ and the PDF is

x	2	3	4	5	6	7	8	9	10	11	12
$P_X(x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$



Joint PDF: More Than One Random Variable

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- Consider again the variables X and Y
  - Define a variable X, to count number of heads (e.g., number-of-heads(HTT) = 1)
  - Define a variable Y, which is 1 if all tosses match (all-tosses-match(HTT) = 0)

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathbb{P}[\omega]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X(\omega)$	3	2	2	1	2	1	1	0
$Y(\omega)$	1	0	0	0	0	0	0	1

- What is $\mathbb{P}[X = 0, Y = 0]$?

$$0$$

- What is $\mathbb{P}[X = 1, Y = 0]$?

$$\frac{3}{8}$$

- Can now define the joint function $P_{XY}(x, y) = \mathbb{P}[X = x, Y = y]$

Joint PDF: More Than One Random Variable, cont'd

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathbb{P}[\omega]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X(\omega)$	3	2	2	1	2	1	1	0
$Y(\omega)$	1	0	0	0	0	0	0	1

- Can now define the joint function $P_{XY}(x, y) = \mathbb{P}[X = x, Y = y]$

$P_{XY}(x, y)$

$P_{XY}(x,y)$	$X = 0$	$X = 1$	$X = 2$	$X = 3$	row sums
$Y = 0$	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
$Y = 1$	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums	$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_Y(y) = \sum_{x \in X(\Omega)} P_{XY}(x, y)$$

$$P_X(x) = \sum_{y \in Y(\Omega)} P_{XY}(x, y)$$

Joint PDF: More Than One Random Variable, cont'd

ω	HHH	HHT	HTH	HTT	THH	THT	TTH	TTT
$\mathbb{P}[\omega]$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X(\omega)$	3	2	2	1	2	1	1	0
$Y(\omega)$	1	0	0	0	0	0	0	1

- Can also compute other events

$$\mathbb{P}[X+Y \leq 2] = 0 + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} + 0 = \frac{7}{8}$$

$$\mathbb{P}[Y = 1 \; and \; X+Y \leq 2] = \frac{1}{8} + 0 = \frac{1}{8}$$

$$\mathbb{P}[Y = 1 | X+Y \leq 2] = \frac{\mathbb{P}[Y = 1 \; and \; X+Y \leq 2]}{\mathbb{P}[X+Y \leq 2]} = \frac{1}{8} \div \frac{7}{8} = \frac{1}{7}$$

$P_{XY}(x,y)$

Y \ X	0	1	2	3	row sums
0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
column sums	$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$$P_Y(y) = \sum_{x \in X(\Omega)} P_{XY}(x,y)$$

$$P_X(x) = \sum_{y \in Y(\Omega)} P_{XY}(x,y)$$

Independent Random Variables

- **Independent Random Variables** measure unrelated quantities. The joint-PDF is *always* the product of the marginals.

$$P_{XY}(x,y) = P_X(x)P_Y(y)\ \ \forall(x,y) \in X(\Omega) \times Y(\Omega)$$

- Our X and Y are **not** independent

$P_{XY}(x,y)$

Y \ X	0	1	2	3	
0	0	$\frac{3}{8}$	$\frac{3}{8}$	0	$\mathbf{\frac{3}{4}}$
1	$\frac{1}{8}$	0	0	$\frac{1}{8}$	$\mathbf{\frac{1}{4}}$
	$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

$P_X(x)P_Y(y)$

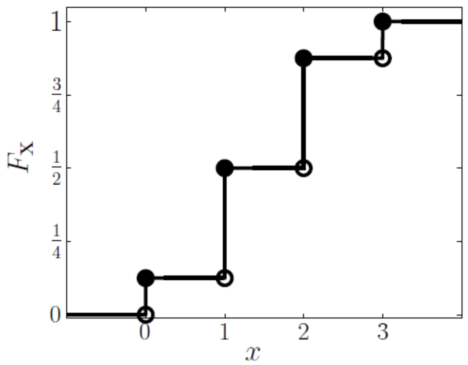
Y \ X	0	1	2	3	
0	$\frac{3}{32}$	$\frac{9}{32}$	$\frac{9}{32}$	$\frac{3}{32}$	$\mathbf{\frac{3}{4}}$
1	$\frac{1}{32}$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$\mathbf{\frac{1}{4}}$
	$\mathbf{\frac{1}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{3}{8}}$	$\mathbf{\frac{1}{8}}$	

- **Practice:** Exercise 18.4, Pop Quizzes 18.5, 18.6.

Cumulative Distribution Function (CDF)

- We are also interested in the probability that a variable is less than a certain number

x	0	1	2	3
$P_X[x]$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
$\mathbb{P}[X \leq x]$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$



- **Cumulative Distribution Function (CDF).** The cumulative distribution function $F_X(x)$ is the probability for the random variable X to be at most x,

$$F_X(x) = \mathbb{P}[X \leq x]$$

Bernoulli Random Variable: Binary Measurable (0,1)

- Bernoulli distributions used to model settings with two possible outcomes
  - e.g., coin tosses, random walks
- The Bernoulli random variable is a binary variable that *indicates* which outcome:

$$X = \begin{cases} 1 & with\ probability & p \\ 0 & with\ probability & 1-p \end{cases}$$

- We can add Bernoullis. Toss n independent coins. X is the number of H:

$$X = X_1 + X_2 + \cdots + X_n$$

  - The variable X is a sum of Bernoullis, each X_i is an independent Bernoulli
- Suppose I make n steps during my random walk. Let R be the number of right steps:

$$R = X_1 + X_2 + \cdots + X_n$$

  - (Assuming a right/left step corresponds to a value of 1/0)
- Similarly, the number of left steps is:

$$L = (1 - X_1) + (1 - X_2) + \cdots + (1 - X_n)$$

- The final position is:

$$X = R - L = 2R - n = 2(X_1 + \cdots + X_n) - n$$

Uniform Random Variable: Every Value Equally Likely

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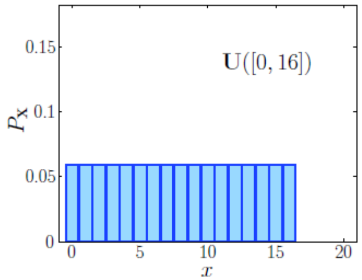
- Suppose a random variable can take on n possible values, $\{1,2,\ldots,n\}$
  - Each with probability $\frac{1}{n}$

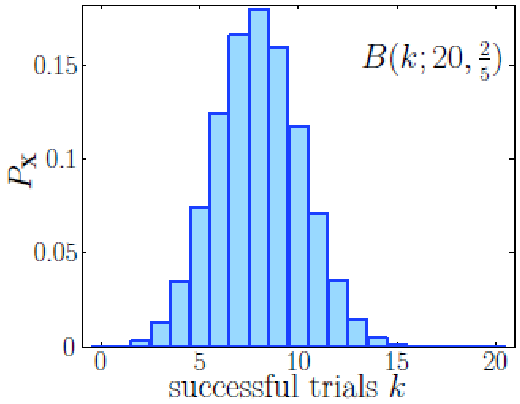
$$P_X(k) = \frac{1}{n}, \quad \text{for } k = 1,\ldots,n$$



- For example, the roll of a 6-sided *fair* die is uniform on $\{1,\ldots,6\}$
  - Written $X \sim U[6]$, where X is the fair die
- Roulette outcomes are uniform on $\{00,0,1,\ldots,36\}$
  - Can remap to $\{1,\ldots,38\}$

Binomial Random Variable: Sum of Bernoullis

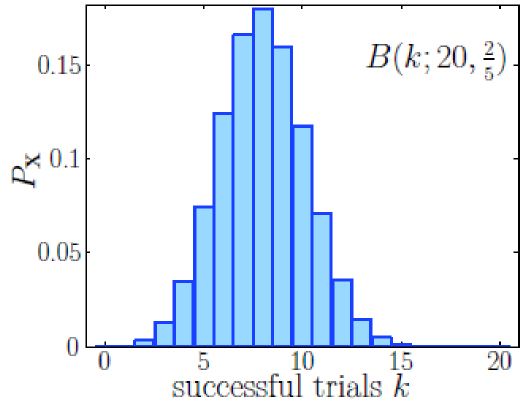
- Let X be the sum of n Bernoulli random variables
  - e.g., X = number of heads in n independent coin tosses with probability p of heads:
  $$X = X_1 + \cdots + X_n$$
  - sum of n independent Bernoullis, $X_i \sim Bernoulli(p)$
- Suppose $n = 5$. What is the probability I get 3 Hs?
  - All 10 outcomes are: {HHHTT, HHTTH, HTTHH, TTHHH, HHTHT, HTHTH, THTHH, HTHHT, THHTH, THHHT}
    - each independently with probability $p^3(1-p)^2$
    $$\mathbb{P}[X = 3 | n = 5] = 10p^3(1-p)^2$$
    - (add outcome probabilities)
- In general, how many outcomes with k Hs are there?
  $$\binom{n}{k}$$
  - each with probability $p^k(1-p)^{n-k}$, so
  $$\mathbb{P}[X = k | n] = \binom{n}{k} p^k (1-p)^{n-k}$$



Binomial Distribution

- **Binomial Distribution. X** is the number of successes in n independent trials with success probability p on each trial: $X = X_1 + \cdots + X_n$, where $X_i \sim Bernoulli(p)$

$$P_X(k) = \mathbb{P}[X = k|n] = B(k; n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

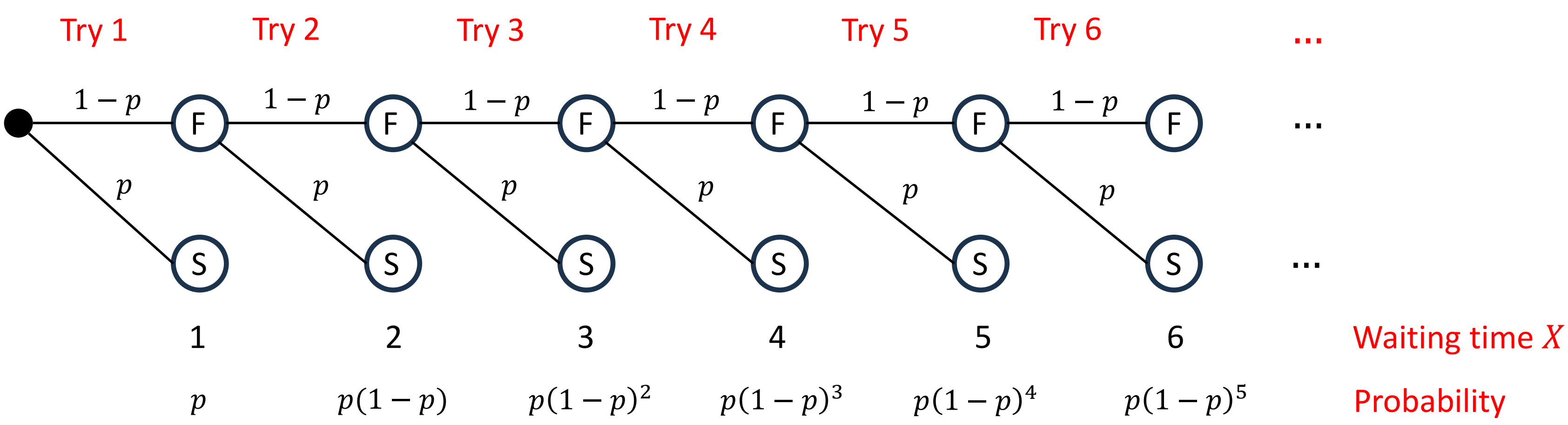


- Example: guessing correctly on the multiple choice quiz: $n = 15$ questions, 5 choices ($p = \frac{1}{5}$).

number correct, k	0	1	2	3	4	5	6	7	8	9	10	11	12	13
probability	0.035	0.132	0.231	0.250	0.188	0.103	0.043	0.014	0.003	$< 10^{-3}$	10^{-4}	10^{-5}	10^{-6}	~ 0

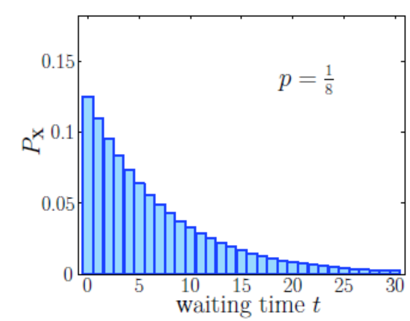
Exponential Random Variable: Waiting Time to Success

- Suppose that you are waiting for a coin to flip H
  - Or waiting for a packet to be successfully transmitted
  - Let p be the probability to succeed on a trial



$$\mathbb{P}[t\ trials] = \mathbb{P}\left[F^{\bullet t-1}S\right] = (1-p)^{t-1}p$$

$$P_X(t) = (1-p)^{t-1}p = \frac{p}{1-p} \times (1-p)^t$$



- Let's rename $\beta = \frac{p}{1-p}$

$$P_X(t) = \beta \times (1-p)^t$$

Exponential Random Variable Example

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- **Example**. 3 people randomly access the wireless channel. A packet is transmitted (i.e., success) only if exactly one is attempting.
  - How do we make sure we make progress?
  - Suppose everyone tries every timestep
    - No one succeeds because there is a collision every time.
  - Suppose everyone tries $\frac{1}{3}$ of the time (randomly)
  - Success probability for *someone* is $3 \times \frac{4}{27} = \frac{4}{9}$
  - Success probability for *you* is $\frac{4}{27}$

wait, t	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$\mathbb{P}$[someone succeeds]	0.444	0.247	0.137	0.076	0.042	0.024	0.013	0.007	0.004	0.002	0.001	$\cdots$
$\mathbb{P}$[you succeed]	0.148	0.126	0.108	0.092	0.078	0.066	0.057	0.048	0.051	0.035	0.030	$\cdots$