Conditional Probability

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 16

Overview

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- New information changes a probability
- Definition of conditional probability from regular probability
- Conditional probability traps
  - Sampling bias
  - Transposed conditional
- Law of total probability
  - Probabilistic case-by-case analysis

Flu Season

- Suppose I told you that, in general, chances a random person has the flu is about 0.01 (or 1%) (*prior* probability).
- Probability of flu:

$$\mathbb{P}[flu] \approx 0.01$$

- Suppose now that you also have a slight fever
  - This is *new information*. Chances of flu “increase”.
- Suppose I know that the probability of flu *given* fever:

$$\mathbb{P}[flu \mid fever] \approx 0.4$$

  - New information changes the prior probability to the *posterior* probability.
  - Translate posterior as “*After* you get the new information.”
- $\mathbb{P}[A|B]$ is the (updated) *conditional* probability of A, *given* the new information B
- Suppose also your roommate has flu (more new information).
  - Flu almost surely!
- Probability of flu *given* fever and roommate flu:

$$\mathbb{P}[flu|fever\ and\ rommate\ flu] \approx 1$$

CS, MATH and Dual CS-MATH Majors

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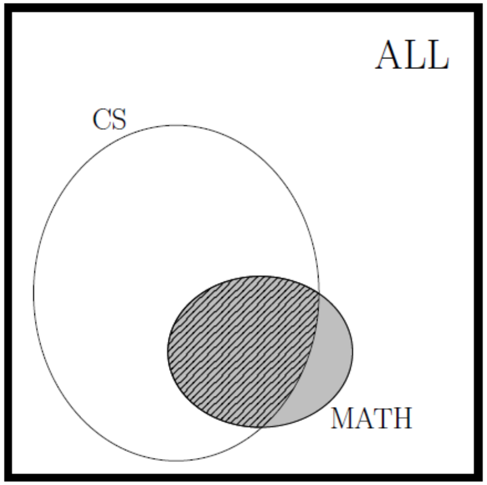

- Suppose RPI has 5000 students total, with 1000 in CS (vast underestimate!), 100 in MATH, 80 dual CS-MATH
- If you pick a random student from RPI, what's the chance the student is in CS

$$\mathbb{P}[CS] = \frac{1000}{5000} = 0.2$$

$$\mathbb{P}[MATH] = \frac{100}{5000} = 0.02$$

$$\mathbb{P}[CS \text{ and } MATH] = \frac{80}{5000} = 0.016$$

- Suppose (after you picked the student), I told you the student is a MATH major
  - New information. What is $\mathbb{P}[CS|MATH]$?
    - Effectively picking a random student from MATH
    - New probability of CS ~ striped area $|CS \cap MATH|$

$$\mathbb{P}[CS|MATH] = \frac{\mathbb{P}[CS \cap MATH]}{\mathbb{P}[MATH]} = \frac{80}{100} = 0.8$$

  - MATH majors 4 times more likely to be CS majors than a random student.
- **Exercise 16.2.**

Conditional Probability $\mathbb{P}[A|B]$

- Conditional probability is interpreted as follows:

$\mathbb{P}[A|B]$ = frequency of outcomes known to be in B that are also in A

- Suppose event B contains n_B outcomes when you repeat an experiment *n* times:

$$\mathbb{P}[B] = \frac{n_B}{n}$$

  – Of the n_B outcomes in B, the number also in A is $n_{A\cap B}$

$$\mathbb{P}[A \cap B] = \frac{n_{A\cap B}}{n}$$

- The frequency of outcomes in A among those outcomes in B is $n_{A\cap B}/n_B$

$$\mathbb{P}[A|B] = \frac{n_{A\cap B}}{n_B} = \frac{n_{A\cap B}}{n} \times \frac{n}{n_B} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

- The definition of conditional probability is then:

$$\mathbb{P}[A|B] = \frac{n_{A\cap B}}{n_B} = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]} = \frac{\mathbb{P}[A\ and\ B]}{\mathbb{P}[B]}$$

Chance of Rain Given Clouds

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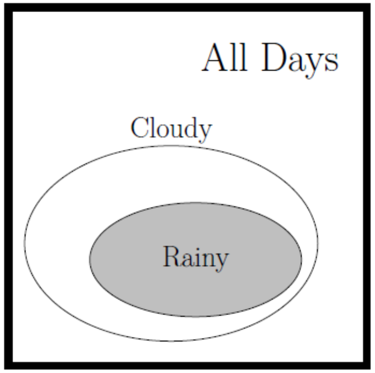


- It is cloudy one in five days, $\mathbb{P}[Clouds] = \frac{1}{5}$
- It rains one in seven days, $\mathbb{P}[Rain] = \frac{1}{7}$
- What are the chances of rain on a cloudy day?
  - Trick question!

$$\mathbb{P}[Rain|Clouds] = \frac{\mathbb{P}[Rain \cap Clouds]}{\mathbb{P}[Clouds]}$$

  - We need to know $\mathbb{P}[Rain \cap Clouds]$
  - It is mostly safe to assume that $\{Rainy\ Days\} \subseteq \{Cloudy\ Days\}$
  - This means that $\mathbb{P}[Rain \cap Clouds] = \mathbb{P}[Rain]$

$$\mathbb{P}[Rain|Clouds] = \frac{\mathbb{P}[Rain \cap Clouds]}{\mathbb{P}[Clouds]} = \frac{\frac{1}{7}}{\frac{1}{5}} = \frac{5}{7}$$

- 5-times more likely to rain on a cloudy day than on a random day.
- Crucial first step: identify the conditional probability. What is the “new information”?

Dicey Conditions

- Here's an odd question:

$$\mathbb{P}[Sum\ of\ 2\ dice\ is\ 10 | Both\ are\ Odd]$$

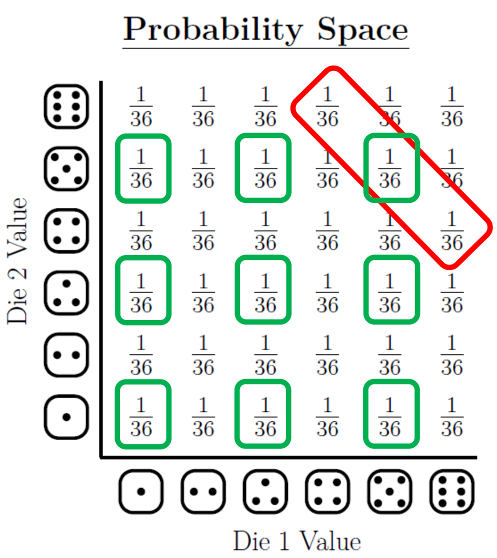
  - Two dice have both rolled odd. What are the chances the sum is 10?
- First, write the definition of conditional probability

$$\mathbb{P}[Sum\ is\ 10 | Both\ are\ Odd] = \frac{\mathbb{P}[(Sum\ is\ 10)\ AND\ (Both\ are\ Odd)]}{\mathbb{P}[Both\ are\ Odd]}$$

- What is the probability space?
  - Let's get counting!

$$\mathbb{P}[Sum\ is\ 10] = \frac{3}{36} = \frac{1}{12}$$

$$\mathbb{P}[Both\ are\ Odd] = \frac{9}{36} = \frac{1}{4}$$



Dicey Conditions

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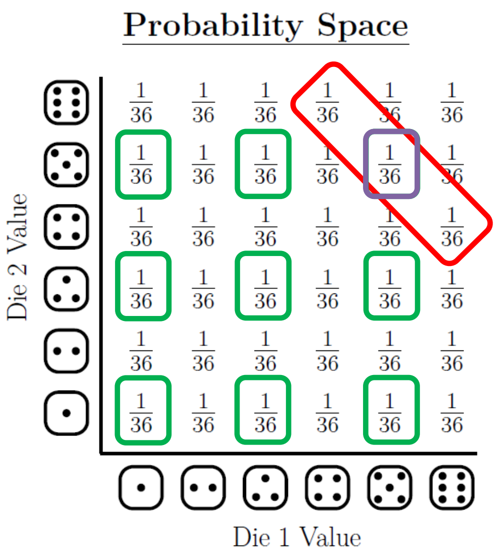

$$\mathbb{P}[Sum\ is\ 10] = \frac{3}{36} = \frac{1}{12}$$

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$$\mathbb{P}[(Sum\ is\ 10)\ AND\ (Both\ are\ Odd)] = \frac{1}{36}$$

$$\mathbb{P}[Sum\ is\ 10 | Both\ are\ Odd] = \frac{1}{36} \div \frac{1}{4} = \frac{1}{9}$$

Computing a Conditional Probability

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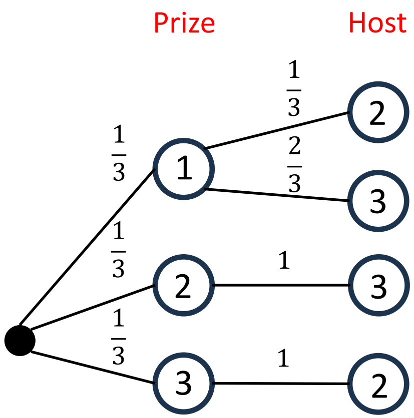
1. Identify that you need a conditional probability $\mathbb{P}[A|B]$
2. Determine the probability space $(\Omega, \mathbb{P})$ using the outcome-tree method
3. Identify the events A and B appearing in $\mathbb{P}[A|B]$ as subsets of Ω
4. Compute $\mathbb{P}[A \cap B]$ and $\mathbb{P}[B]$
5. Compute $\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$

Monty Prefers Door 3

- Recall the Monty Hall outcome tree
  - Suppose Monty prefers door 3
- What is the best strategy now?
  - You should still switch
  - Winning outcomes are the same!

$$\mathbb{P}[WinBySwitching] = \frac{2}{3}$$



Outcome	Probability
(1,2)	$\mathbb{P}[(1,2)] = \frac{1}{9}$
(1,3)	$\mathbb{P}[(1,3)] = \frac{2}{9}$
(2,3)	$\mathbb{P}[(2,3)] = \frac{1}{3}$
(3,2)	$\mathbb{P}[(3,2)] = \frac{1}{3}$

- Interestingly, you should feel better if you see Monty opening door 2!
  - Intuitively, why is this true?
  - Monty prefers door 3, so if he opened 2, then 3 very likely contains the prize!

$$\mathbb{P}[Win|Monty\ Opens\ Door\ 2] = \frac{\mathbb{P}[Win\ AND\ Monty\ Opens\ Door\ 2]}{\mathbb{P}[Monty\ Opens\ Door\ 2]}$$

$$= \frac{\frac{1}{3}}{\frac{1}{9} + \frac{1}{3}} = \frac{3}{4}$$

- Your chances improved from $\frac{2}{3}$ to $\frac{3}{4}$!

A Pair of Slides

- Suppose I have to make 2 lecture slides in 2 minutes
- The probability of having a typo on a slide is 0.5
  - What is the probability that both slides have typos?
  - Answer: $\frac{1}{4}$
- What is the probability that both slides have typos given the additional information in each of the cases below?
- At least one slide contains a typo
  - Answer: $1/3$

$$\mathbb{P}[2\ typos|at\ least\ 1\ typo] = \frac{\mathbb{P}[2\ typos\ AND\ at\ least\ 1\ typo]}{\mathbb{P}[at\ least\ 1\ typo]} = \frac{\frac{1}{4}}{\frac{3}{4}}$$

- The second slide has a typo
  - Answer: $1/2$

$$\mathbb{P}[2\ typos|second\ slide\ has\ a\ typo] = \frac{\mathbb{P}[2\ typos\ AND\ second\ slide\ has\ a\ typo]}{\mathbb{P}[second\ slide\ has\ a\ typo]} = \frac{\frac{1}{4}}{\frac{1}{2}}$$

- I made one of the slides during office hours and made a typo
  - Answer: $1/2$

$$\mathbb{P}[2\ typos|1\ typo\ during\ OH] = \frac{\mathbb{P}[2\ typos\ AND\ 1\ typo\ during\ OH]}{\mathbb{P}[1\ typo\ during\ OH]} = \frac{\frac{1}{4}}{\frac{1}{2}}$$

- I wanted to illustrate my typo-making process, so I showed one slide with a typo
  - Answer: $1/3$

$$\mathbb{P}[2\ typos|at\ least\ 1\ typo] = \frac{\mathbb{P}[2\ typos\ AND\ at\ least\ 1\ typo]}{\mathbb{P}[at\ least\ 1\ typo]} = \frac{\frac{1}{4}}{\frac{3}{4}}$$

- Same question in each case, but with slightly different additional information.

Conditional Probability Traps

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- These four probabilities are all different

$$\mathbb{P}[A] \qquad \mathbb{P}[A|B] \qquad \mathbb{P}[B|A] \qquad \mathbb{P}[A \cap B]$$

- Be careful which one you use!!

- **Sampling bias**: Using $\mathbb{P}[A]$ instead of $\mathbb{P}[A|B]$
  - Pollsters have a very tricky job when conducting surveys of public opinion
  - How do they contact people (landline, cell phone, social media, snail mail?)
  - Suppose you ask people "What is the probability that AI goes rogue?" and the only responses you get are through mail
  - What value are you surveying?

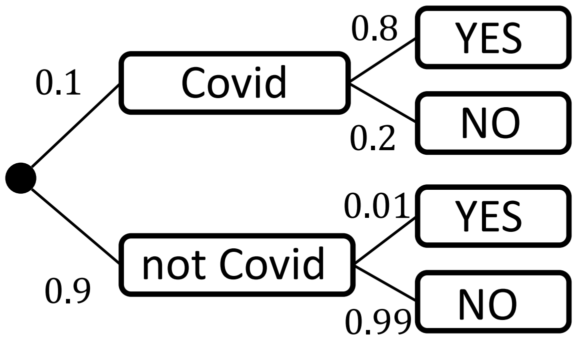
$$\mathbb{P}[rogue\ AI|responder\ uses\ mail]$$

  - This number is likely very different from $\mathbb{P}[rogue\ AI]$
- Transposed conditional: using $\mathbb{P}[B|A]$ instead of $\mathbb{P}[A|B]$
  - See book for the famous Lombard study

The Covid Test and Transposed Conditionals

- At-home antigen covid tests have different accuracy depending on the case
  - If you have covid, the test will make a mistake ~20% of the time
  - If you don't have covid, the test will make a mistake ~1% of the time
  - Source: https://www.cochrane.org/CD013705/INFECTN_how-accurate-are-rapid-antigen-tests-diagnosing-covid-19
- Suppose you tested positive. What are the chances you actually have covid?
  - If you don't have covid, the test is unlikely to be wrong, so you are likely sick
- It is wrong to look at $\mathbb{P}[positive|not\ Covid]$. We already have this probability!
  - We need $\mathbb{P}[not\ Covid|positive]$

Covid | Covid test | Probability



$\mathbb{P}[Covid, YES] = 0.1 \times 0.8$

$\mathbb{P}[Covid, NO] = 0.1 \times 0.2$

$\mathbb{P}[not\ Covid, YES] = 0.9 \times 0.01$

$\mathbb{P}[not\ Covid, NO] = 0.9 \times 0.99$

The Covid Test and Transposed Conditionals

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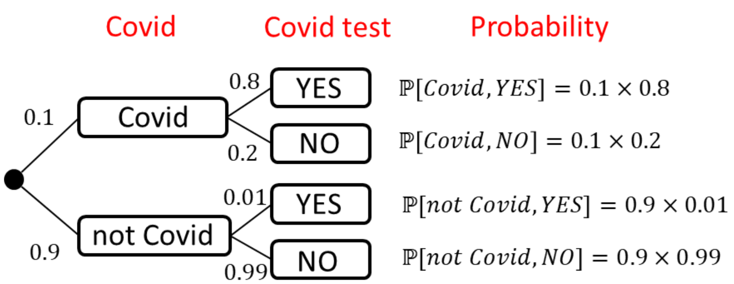
- It is wrong to look at $\mathbb{P}[positive|not\ Covid]$. We already have this probability!
  - We need $\mathbb{P}[not\ Covid|positive]$

$$\mathbb{P}[not\ Covid|positive] = \frac{\mathbb{P}[not\ Covid\ AND\ YES]}{\mathbb{P}[YES]}$$
$$= \frac{0.9 \times 0.01}{0.1 \times 0.8 + 0.9 \times 0.01}$$
$$\approx 10\%$$

- The less accurate test says YES but the chances are 90% that you have covid
  - Two possibilities:
  - You don't have covid (likely) and test made a mistake (very rare)
  - You have covid (rare) and test was correct (likely). Wins!



Total Probability: Case by Case Probability

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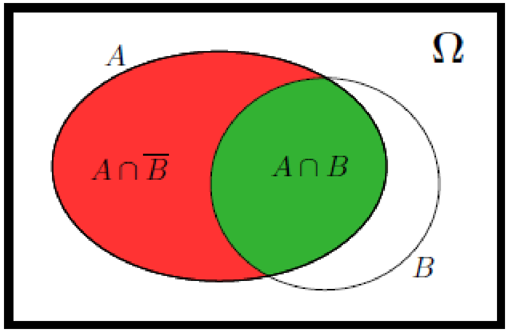


- Two types of outcomes in any event A:
  - The outcomes in B (green)
  - The outcomes not in B (red)

$$\mathbb{P}[A] = \mathbb{P}[A \cap B] + \mathbb{P}[A \cap \bar{B}] \quad (*)$$

    - (Similar to sum rule from counting)
    - (Also known as marginalization)

- From the definition of conditional probability:

$$\mathbb{P}[A \cap B] = \mathbb{P}[A \text{ AND } B] = \mathbb{P}[A|B] \times \mathbb{P}[B]$$
$$\mathbb{P}[A \cap \bar{B}] = \mathbb{P}[A \text{ AND } \bar{B}] = \mathbb{P}[A|\bar{B}] \times \mathbb{P}[\bar{B}]$$

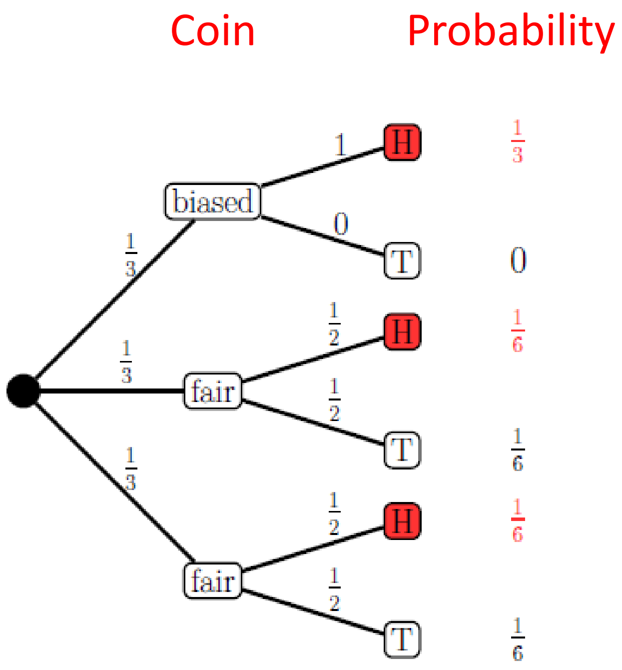
- Plugging these in $(*)$, we get a **FUNDAMENTAL** result for case-by-case analysis
- **LAW OF TOTAL PROBABILITY**:

$$\mathbb{P}[A] = \mathbb{P}[A|B] \times \mathbb{P}[B] + \mathbb{P}[A|\bar{B}] \times \mathbb{P}[\bar{B}]$$

- (Weight conditional probabilities for each case by probabilities of each case and add.)

Three Coins: Two Are Fair, One is 2-Headed

- Pick a random coin and flip. What is the probability of H?
- First, let's use the outcome-tree method as before



- What is the probability of H?

$$\mathbb{P}[H] = \frac{1}{3} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$$

Three Coins: Two Are Fair, One is 2-Headed, cont'd

- Pick a random coin and flip. What is the probability of H?
- Now, let's use the law of total probability:
  - Case 1. B: You picked one of the fair coins
  - Case 2. $\bar{B}$: You picked the two-headed coin

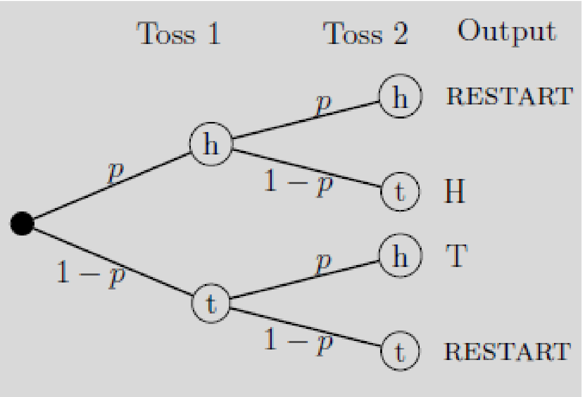
$$\mathbb{P}[H] = \mathbb{P}[H|B] \times \mathbb{P}[B] + \mathbb{P}\left[H\middle|\bar{B}\right] \times \mathbb{P}\left[\bar{B}\right]$$
$$= \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{2}{3}$$

- Note that we don't have to draw the (potentially exponentially growing) outcome tree anymore
- **Exercise.** A box has 10 coins: 6 fair and 4 biased (probability of heads $\frac{2}{3}$). What is $\mathbb{P}[2\ heads]$ in each case?
  - Pick a single random coin and flip it 3 times.
  - Flip 3 times. For each flip, pick a random coin, flip it and then put the coin back.

Fair Toss from Biased Coin (*unknown* probability p of heads)?

- Suppose I would like to use a biased coin in order to obtain a fair outcome
  - The challenge is that I don't even know the probability of H or T for this coin
  - How do I do it?
  - One option would be to toss 10^{1000} times and estimate p
  - A more pragmatic method is to notice that $\mathbb{P}['ht'] = \mathbb{P}['th'] = p(1-p)$
    - (Lower case 'h' and 't' denote the outcomes of a toss.)
  - This suggests that an H is as likely as a T.
  - I can now design the following algorithm:

1. Make two tosses of the biased coin
2. If you get 'ht' output H; 'th' output T; otherwise RESTART.



Fair Toss from Biased Coin (*unknown* probability p of heads)?, cont'd

- Let's use the law of total probability to estimate the value of H output by our algorithm

$$\begin{aligned}\mathbb{P}[H] &= \mathbb{P}[H|RESTART]\cdot\mathbb{P}[RESTART] + \mathbb{P}[H|'ht']\cdot\mathbb{P}['ht'] + \mathbb{P}[H|'th']\cdot\mathbb{P}['th'] \\ &= \mathbb{P}[H]\cdot\left(p^2+(1-p)^2\right) + 1\cdot p(1-p) + 0\cdot p(1-p) \\ &= \mathbb{P}[H]\cdot\left(p^2+(1-p)^2\right) + p(1-p)\end{aligned}$$

- Solve for $\mathbb{P}[H]$:

$$\mathbb{P}[H] = \frac{p(1-p)}{1-(p^2+(1-p)^2)} = \frac{p(1-p)}{2p-2p^2} = \frac{p(1-p)}{2p(1-p)} = \frac{1}{2}$$

- (You can also solve this problem using an infinite outcome tree and computing an infinite sum.)