Probability

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 15

Overview

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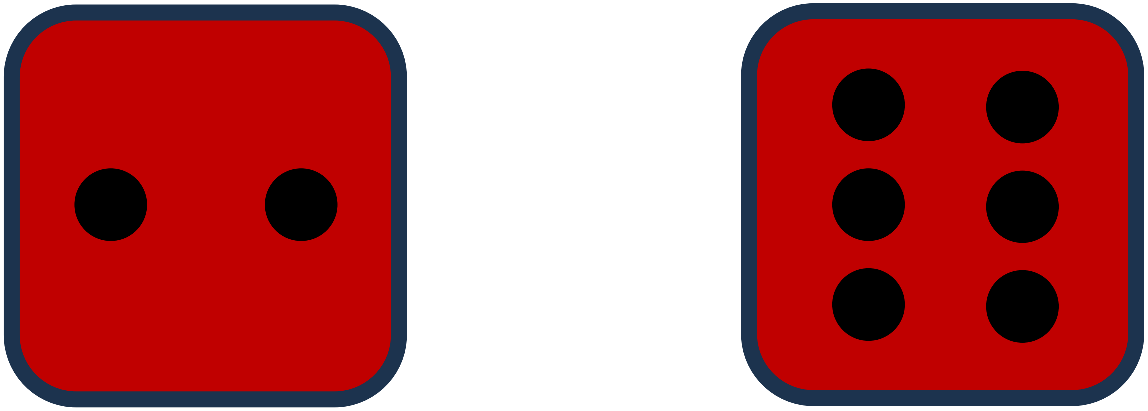
- Computing probabilities
  - Outcome tree
  - Event of interest
  - Examples with dice
- Probability and sets
  - The probability space
- Uniform probability spaces
- Infinite probability spaces

Probability



The Chance of Rain Tomorrow is 40%

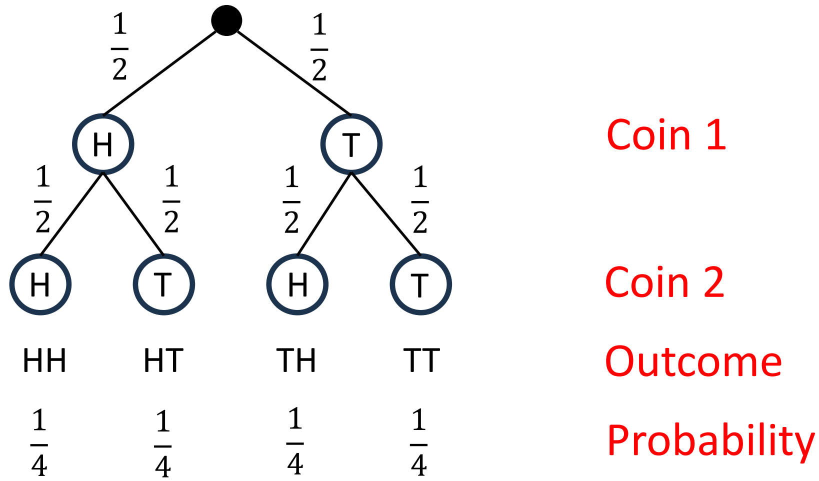
- What does the title mean? Either it will rain tomorrow or it won’t.
  - The chances are 50% that a *fair* coin-flip will be H.
  - Flip 100 times. Approximately 50 will be H
    - This is known as the *frequentist* view.
    - As opposed to the Bayesian view, which comes with a prior assumption about the world
      - e.g., coins are assumed fair unless we have sufficient evidence that they’re not
- Consider the following scenarios
  - You toss a *fair* coin 3 times. How many heads will you get?
  - You keep tossing a *fair* coin until you get a head. How many tosses will you make?
- There’s no answer. The outcome is uncertain. Probability handles such settings.

Birth of Mathematical Probability

- *Antoine Gombaud, Chevalier de Méré*: Should I bet even money on at least one 'double-6' in 24 rolls of two dice? What about at least one 6 in 4 rolls of one die?
- *Blaise Pascal*: Interesting question. Let's bring *Pierre de Fermat* into the conversation.
  - . . . a correspondence is ignited between these two mathematical giants

Toss Two Coins: You Win if the Coins Match (HH or TT)

- You are analyzing an "experiment" whose outcome is uncertain.
- **Outcomes.** Identify all *possible* outcomes using a *tree* of outcome *sequences*

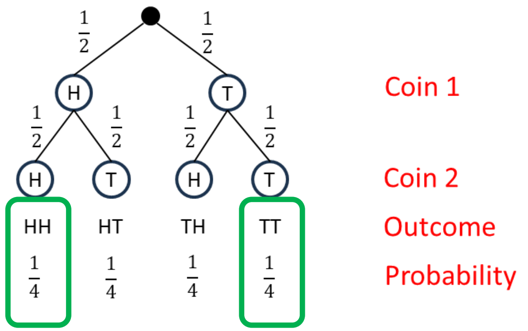


- **Edge probabilities.** If one of k edges (options) from a vertex is chosen *randomly* then what edge-probability does each edge have?

$$\frac{1}{k}$$

- **Outcome-probability.** Multiply edge-probabilities to get outcome-probabilities.

Event of Interest



- Toss two coins: you win if the coins match (HH or TT)
- **Question:** When do you win?
- **Event:** *Subset* of outcomes where you win.
- **Event-probability.** Sum of its outcome-probabilities.

$$\text{event-probability} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

- *Probability* that you win is $\frac{1}{2}$
  - Written $\mathbb{P}["You\ Win"] = \frac{1}{2}$
- Go and do this experiment at home. Toss two coins 1000 times and see how often you win.
  - What do you think is the likelihood of winning 1000 times?
  - Seems very unlikely

The Outcome-Tree Method

- Become familiar with this 6-step process for analyzing a probabilistic experiment.

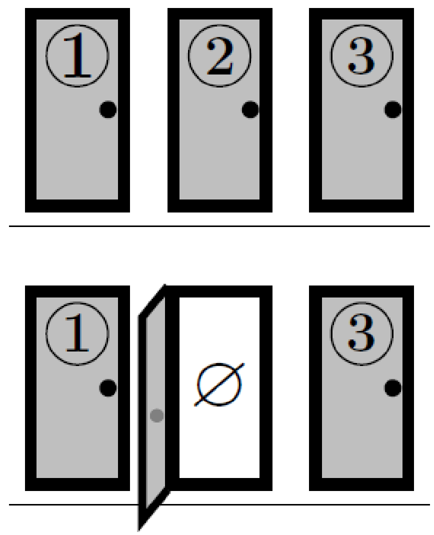
1. You are analyzing an experiment whose outcome is uncertain.
2. **Outcomes.** Identify *all possible* outcomes, the tree of *outcome sequences*.
3. **Edge-Probability.** Each edge in the outcome-tree gets a probability.
4. **Outcome-Probability.** Multiply edge-probabilities to get outcome-probabilities.
5. **Event of Interest** $\mathcal{E}$. Determine the subset of the outcomes you care about.
6. **Event-Probability.** The sum of outcome-probabilities in the subset you care about.

$$\mathbb{P}[\mathcal{E}] = \sum_{outcomes\ \omega\in\mathcal{E}} \mathbb{P}[\omega]$$

   - $\mathbb{P}[\mathcal{E}]$ frequency an outcome you want occurs over many repeated experiments.
   - New notation: $\sum_{outcomes\ \omega\in\mathcal{E}} \mathbb{P}[\omega]$
     - Suppose $\mathcal{E} = \{1,2,3\}$. Then

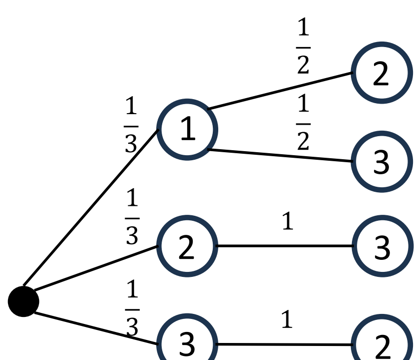
$$\sum_{outcomes\ \omega\in\mathcal{E}} \mathbb{P}[\omega] = \mathbb{P}[1] + \mathbb{P}[2] + \mathbb{P}[3]$$

Let's Make a Deal: The Monty Hall Problem



1. Contestant at door 1.
2. Prize placed behind *random* door.
3. Monty opens empty door (*randomly* if there's an option).

- Outcome-tree and edge-probabilities
- Outcome-probabilities
- Event of interest: "WinBySwitching"
- Event probability

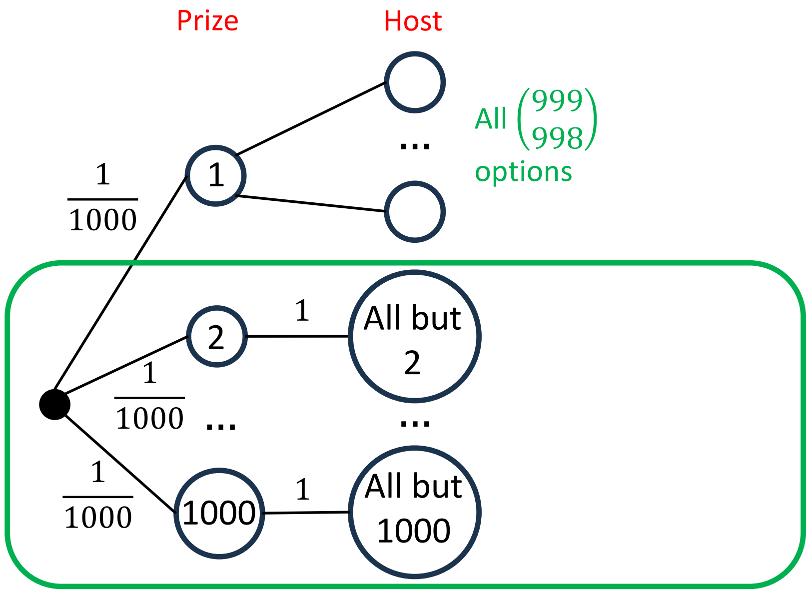


Outcome	Probability
(1,2)	$\mathbb{P}[(1,2)] = \frac{1}{6}$
(1,3)	$\mathbb{P}[(1,3)] = \frac{1}{6}$
(2,3)	$\mathbb{P}[(2,3)] = \frac{1}{3}$
(3,2)	$\mathbb{P}[(3,2)] = \frac{1}{3}$

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3} = \mathbb{P}["WinBySwitching"]$$

Monty Hall, cont'd

- You might be thinking "This is a hoax! Doors are chosen randomly, what is this weird math we're learning?"
- But opened doors actually reveal a lot of information!
- Let's make the probabilities radically obvious
  - Suppose there are 1000 doors (still 1 prize)
    - You pick door 1, and Monty opens 998 wrong doors!
  - Let's look the outcome tree



You win by switching $\frac{999}{1000}$ times!

Non-Transitive 3-Sided Dice

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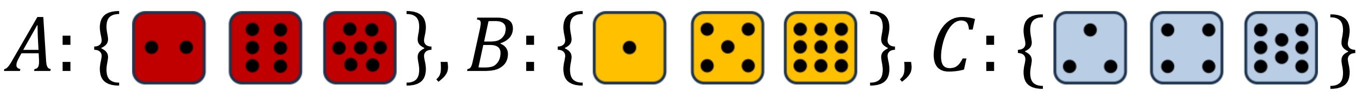
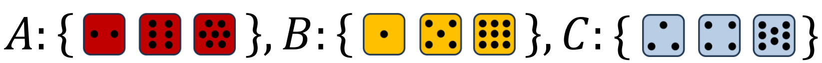
- Consider the following 3 dice



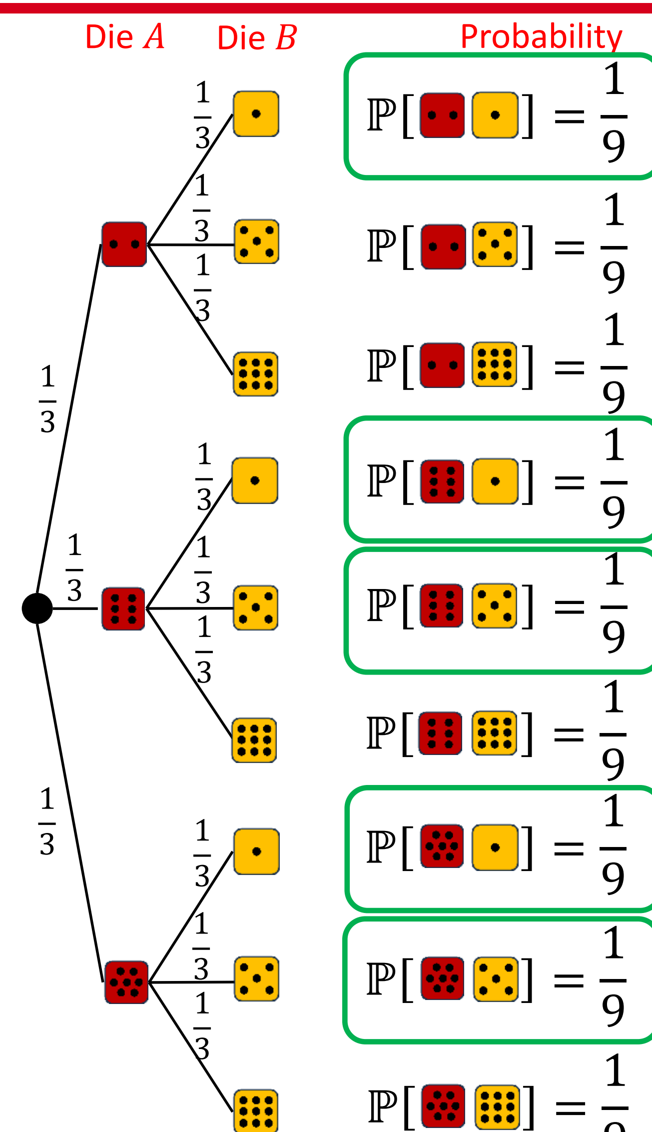
- Notice that A rolls higher than B more than half the time and B rolls higher than C more than half the time
  - But C rolls higher than A more than half the time ! (weird probabilities…)
  - Called non-transitive dice (why?)
    - (transitive means $A \geq B \cap B \geq C \rightarrow A \geq C$)
  - Dice from course 6.042J, ocw.mit.edu. See also Wikipedia, non-transitive dice
- Let's investigate this in detail!
  - Your friend picks a die and then you pick a die.
  - E.g. friend picks B and then you pick A
- **What is the probability that A beats B?**

Non-Transitive 3-Sided Dice, cont'd

- Consider the following 3 dice



- **What is the probability that A beats B?**
- Outcome-tree and outcome-probabilities.
- Uniform probabilities.
- Event of interest: outcomes where A wins
- Number of outcomes where A wins: 5
- $\mathbb{P}[A\ beats\ B] = \frac{5}{9}$
- **Conclusion:** Die A beats Die B



Probability and Sets: The Probability Space

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- **Sample Space** $\Omega = \{\omega_1, \omega_2, \ldots\}$, set of *possible* outcomes
- **Probability Function** $\mathbb{P}$. Non-negative function $\mathbb{P}[\omega]$, normalized to 1:

$$0 \le \mathbb{P}[\omega] \le 1 \quad \text{AND} \quad \sum_{\omega \in \Omega} \mathbb{P}[\omega] = 1$$

(Die A versus B) Ω	$(2,1)$	$(2,5)$	$(2,9)$	$(6,1)$	$(6,5)$	$(6,9)$	$(7,1)$	$(7,5)$	$(7,9)$
$\mathbb{P}(\omega)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

- Events $\mathcal{E} \subseteq \Omega$ are subsets. Event probability $\mathbb{P}[\mathcal{E}]$ is the sum of outcome-probabilities

“$A > B$” $\quad \mathcal{E}_1 = \{(2,1), (6,1), (6,5), (7,5), (7,1)\}$

“Sum > 8” $\quad \mathcal{E}_2 = \{(2,5), (6,5), (6,9), (7,5), (7,9)\}$

“$B < 9$” $\quad \mathcal{E}_3 = \{(2,1), (2,5), (6,1), (6,5), (7,1), (7,5)\}$

- Combining events using logical connectors corresponds to set operations:

“$A > B$ ∨ Sum > 8” $\quad \mathcal{E}_1 \cup \mathcal{E}_2 = \{(2,1), (6,1), (6,5), (6,5), (7,1), (2,5), (6,9), (7,9)\}$

“$A > B$ ∧ Sum > 8” $\quad \mathcal{E}_1 \cap \mathcal{E}_2 = \{(6,5), (7,5)\}$

$\neg$(“$A > B$”) $\quad \overline{\mathcal{E}_1} = \{(2,5), (2,9), (6,9), (7,9)\}$

“$A > B$” → “$B < 9$” $\quad \mathcal{E}_1 \subseteq \mathcal{E}_3$

Important Probability Exercise

- **Exercise 15.10.** Sum rule, complement, inclusion-exclusion, union, implication and intersection bounds.

Uniform Probability Space: Probability ~ Size

- So far, we've mostly looked at uniform probability spaces
  - Each individual outcome has the same probability
  - Fair coin, fair dice, uniform price placement in the Monty Hall Problem
- Formally, each outcome ω has probability

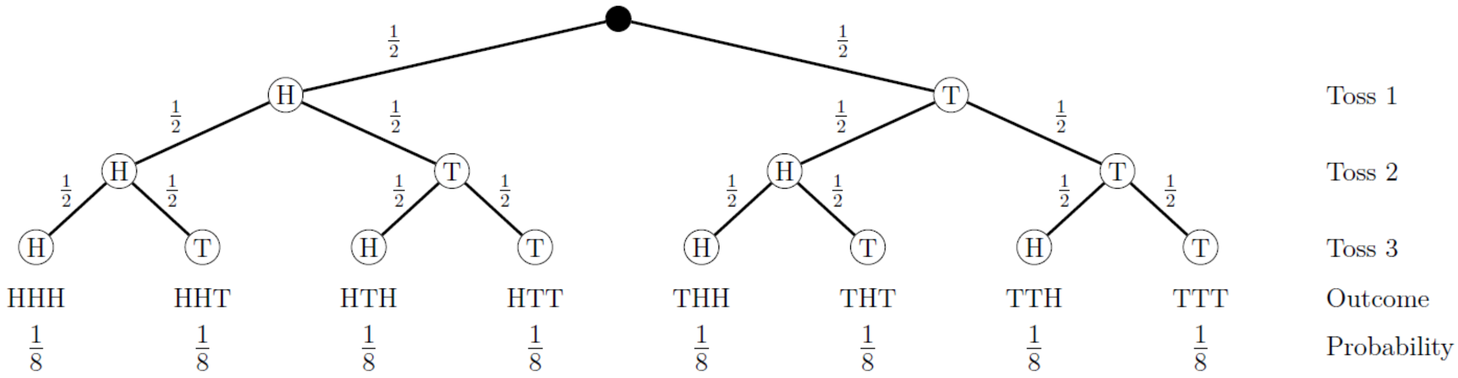
$$\mathbb{P}[\omega] = \frac{1}{|\Omega|}$$

- Each event $\mathcal{E}$ has probability

$$\mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{number\ of\ outcomes\ in\ \mathcal{E}}{number\ of\ possible\ outcomes\ in\ \Omega}$$

Uniform Probability Space Example

- Toss a coin 3 times



- What is $\mathbb{P}[\text{"2 } heads\text{"}]$?

$$\mathbb{P}[2\ heads] = \frac{number\ of\ sequences\ with\ 2\ heads}{number\ of\ possible\ sequences\ in\ \Omega} = \binom{3}{2} \times \frac{1}{8} = \frac{3}{8}$$

- **Practice: Exercise 15.11.**
  - You roll a pair of regular dice. What is the probability that the sum is 9?
  - You toss a fair coin ten times. What is the probability that you obtain 4 heads?
  - You roll die A ten times. Compute probabilities for: 4 sevens? 4 sevens and 3 sixes? 4 sevens or 3 sixes?

Poker: Probabilities of Full House and Flush

- 52-card deck has 4 suits (H,C,S,D) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2)
  - I often wonder: why does full house beat flush?
- Randomly deal 5 cards: each *set* is equally likely → uniform probability space

$$\text{number of possible outcomes} = \binom{52}{5} \text{ possible hands}$$

- **Full house:** 3 cards of one rank and 2 of another.
  - How many full-houses are there?
  - To construct a full house, specify $(rank_3, suit_3, rank_2, suit_2)$.
  - Count all combinations using the product rule:

$$\#\, full\ houses = 13 \times \binom{4}{3} \times 12 \times \binom{4}{2}$$

  - The probability of getting a full house is then:

$$\frac{13 \times \binom{4}{3} \times 12 \times \binom{4}{2}}{\binom{52}{5}} \approx 0.00144$$

Poker: Probabilities of Full House and Flush

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- 52 card deck has 4 suits (H,C,S,D) and 13 ranks in a suit (A,K,Q,J,T,9,8,7,6,5,4,3,2)
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- Randomly deal 5 cards: each *set* is equally likely → uniform probability space

$$\text{number of possible outcomes} = \binom{52}{5} \text{ possible hands}$$

- **Full house:** 3 cards of one rank and 2 of another.
  - The probability of getting a full house is: 0.00144
- **Flush**: 5 cards of same suit.
  - How many flushes are there?
  - To construct a flush, specify $(suit, ranks)$. Using the product rule:

$$\#\ flushes = 4 \times \binom{13}{5}$$

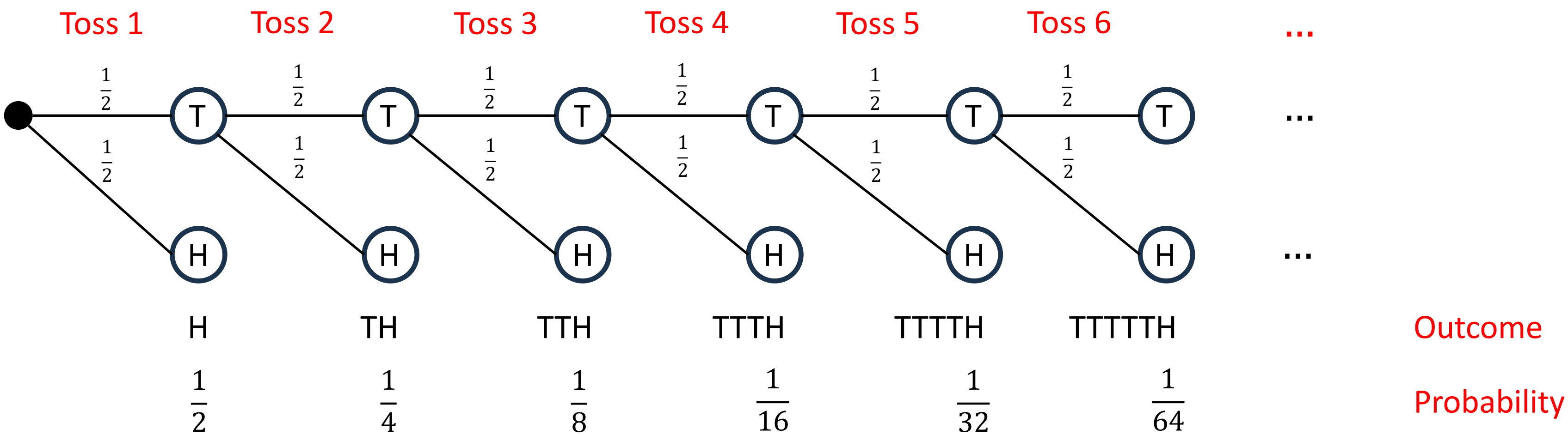
  - The probability of getting a flush is then:

$$\frac{4 \times \binom{13}{5}}{\binom{52}{5}} \approx 0.00198$$

Full house is rarer! That's why full house beats flush!

Toss a Coin Until Heads: Infinite Probability Space

- Suppose you would like to know how many tosses it will take until you get a H



- Hm, seems like the probabilities are halved after each additional T

Ω	H	TH	$T^{\bullet 2}H$	$T^{\bullet 3}H$	$T^{\bullet 4}H$	...	$T^{\bullet i}H$
$P(\omega)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	...	$\left(\frac{1}{2}\right)^{i+1}$
# Tosses	1	2	3	4	5	...	i

Toss a Coin Until Heads: Infinite Probability Space

- Hm, seems like the probabilities are halved after each additional T

Ω	H	TH	$T^{\bullet 2}H$	$T^{\bullet 3}H$	$T^{\bullet 4}H$	…	$T^{\bullet i}H$
$P(\omega)$	$\frac{1}{2}$	$\left(\frac{1}{2}\right)^2$	$\left(\frac{1}{2}\right)^3$	$\left(\frac{1}{2}\right)^4$	$\left(\frac{1}{2}\right)^5$	…	$\left(\frac{1}{2}\right)^{i+1}$
# Tosses	1	2	3	4	5	…	i

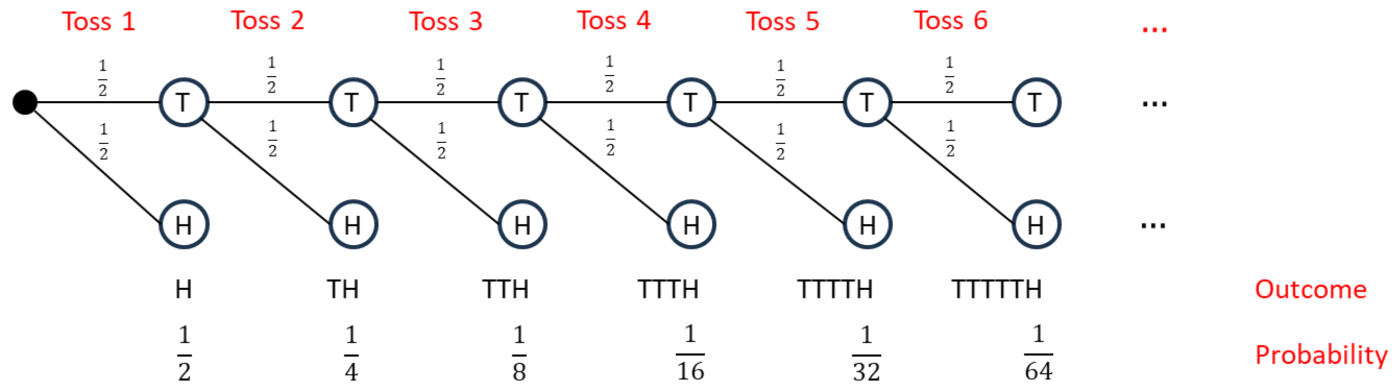
- Sum of outcome probabilities (for sanity checking purposes):

$$\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\cdots=$$

$$=\frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{2}\right)^i$$

$$=\frac{1}{2}\frac{1}{1-\frac{1}{2}}=1$$

Game: First Person to Toss H Wins. Always Go First!



- Look at the relevant outcomes in the table if you go first:

Ω	$\boldsymbol{H}$	TH	$\boldsymbol{T^{\bullet 2}H}$	$T^{\bullet 3}H$	$\boldsymbol{T^{\bullet 4}H}$	...	$T^{\bullet i}H$
$P(\omega)$	$\boldsymbol{\frac{1}{2}}$	$\left(\frac{1}{2}\right)^2$	$\boldsymbol{\left(\frac{1}{2}\right)^3}$	$\left(\frac{1}{2}\right)^4$	$\boldsymbol{\left(\frac{1}{2}\right)^5}$	...	$\left(\frac{1}{2}\right)^{i+1}$

- The event "YouWin" is $\mathcal{E} = \{H, T^{\bullet 2}H, T^{\bullet 4}H, \ldots\}$

$$\mathbb{P}[YouWin] = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots = \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{4}\right)^i =$$

$$= \frac{1}{2}\frac{1}{1-\frac{1}{4}} = \frac{2}{3}$$

  – Your odds improve by a factor of 2 if you go first (vs. second).