Counting

Reading

- Malik Magdon-Ismail. Discrete Mathematics and Computing.
  - Chapter 13

Overview

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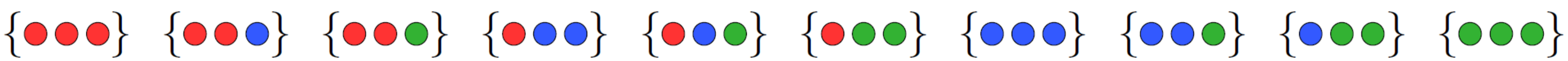
- Counting sequences.
- Build-up counting.
- Counting one set by counting another: bijection.
- Permutations and combinations.

Discrete Math is About Objects We Can Count

- Three colors of candy: red, blue, green
- A goody-bag has 3 candies. How many distinct goody-bags?
- (Only the number of each color matters: bags with different orderings are the "same" goody-bag.)



- **Challenge Problems.**
  - What if there are 5 candies per goody-bag and 10 colors of candy?
  - Goody-bags come in bulk packs of 5. How many different bulk packs are there?
- There are too many to list out. We need tools!

Sum Rule

- How many binary sequences of length 3

$$\{000, 001, 010, 011, 100, 101, 110, 111\}$$

- There are two types: those ending in 0 and those ending in 1,

$$\{b_1 b_2 b_3\} = \{b_1 b_2 \bullet 0\} \cup \{b_1 b_2 \bullet 1\}$$

- **Sum Rule.** N objects of two types: N_1 of type-1 and N_2 of type-2. Then,

$$N = N_1 + N_2$$

- Going back to the binary example:

$$|\{b_1 b_2 b_3\}| = |\{b_1 b_2 \bullet 0\}| + |\{b_1 b_2 \bullet 1\}| \qquad \textbf{[sum rule]}$$
$$= |\{b_1 b_2\}| \times 2$$
$$= (|\{b_1 \bullet 0\}| + |\{b_1 \bullet 1\}|) \times 2 \qquad \textbf{[sum rule]}$$
$$= |\{b_1\}| \times 2 \times 2$$
$$= 2 \times 2 \times 2$$

Product Rule

- Number of choices rule
$$|\{b_1b_2b_3\}| = 2 \times 2 \times 2$$
- **Product Rule**. Let N be the number of choices for a sequence
$$x_1x_2x_3 \cdots x_{r-1}x_r$$
  - Let N_1 be the number of choices for x_1;
  - Let N_2 be the number of choices for x_2 *after you choose* x_1;
  - Let N_3 be the number of choices for x_3 *after you choose* x_1x_2;
  - Let N_4 be the number of choices for x_4 *after you choose* $x_1x_2x_3$;
  - ...
  - Let N_r be the number of choices for x_r *after you choose* $x_1x_2x_3 \dots x_{r-1}$
$$N = N_1 \times N_2 \times N_3 \times N_4 \times \cdots \times N_r$$

- **Example.** There are 2^n binary sequences of length n:
$$N_1 = N_2 = \cdots = N_n = 2$$
- The sum and product rules are the only basic tools we need . . . plus **TINKERING**.

Examples

- **Menus**.
  - $breakfast \in \{pancake, waffle, coffee\}$
  - $lunch \in \{burger, coffee\}$
  - $dinner \in \{salad, steak, coffee\}$

$$|\{BLD\}| = 3 \times 2 \times 3 = 18$$

  - (every menu is a sequence *BLD and* every sequence *BLD* is a *unique* menu.)
- **NY Plates**.
  - A NY plate has the form $\{ABC - 1234\}$

$$|\{ABC - 1234\}| = 26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 \approx 176M$$

- **Races**.
  - With 10 runners, how many top-3 finishes?

$$|\{FST\}| = 10 \times 9 \times 8 = 720$$

Examples, cont'd

- **Passwords.**
  - Use: $\{a, \ldots, z\}, \{A, \ldots, Z\}, \{0, \ldots, 9\}$, special: $\{!, @, \#, \$, \%, \wedge, \&, *, (,)\}$
  - Rules: Length is 8. Must have at least one special.
  - Total number is the sum of valid and invalid (no special symbol) passwords

$$|\{passwords\}| = 72 \times 72 \times \cdots \times 72 = 72^8 \qquad \text{[product rule]}$$

$$= |\{valid\}| + |\{invalid\}| \qquad \text{[sum rule]}$$

$$= |\{valid\}| + 62^8 \qquad \text{[product rule]}$$

$$|\{valid\}| = 72^8 - 62^8 \approx 5 \times 10^{14}$$

- (1 millisecond to test → about 6 months on 32K cores.)

- **Committees.**
  - We have 10 students. How many ways to form a party planning committee?
  - Each student can be in or out of the committee:
    - e.g. $\{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\}$
    - Then $1\ 1\ 0\ 1\ 0\ 0\ 0\ 0\ 1\ 0 \leftrightarrow \{s_1, s_2, s_4, s_9\}$, $0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \leftrightarrow \emptyset$
  - Aha: $|\{committees\}| = |\{10 - bit\ binary\ strings\}|$

$$2 \times 2 \times \cdots \times 2 = 2^{10} = 1024$$

Build-up Counting

- Already saw that the total number of binary sequences of length n is 2^n
- How about the number binary sequences of length n with exactly k 1's, $0 \leq k \leq n$?
  - Denote this number by $\binom{n}{k}$
- First, tinker!
  - Length-3 sequences: $000, 001, 010, 011, 100, 101, 110, 111$

k (columns), n (rows)

$\binom{n}{k}$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4									
5									
6									
7									
8									

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  - Length-3 sequences:
    000,001,010,011,100,101,110,111
  - Legnth-4 sequences:
    0000,0001,0010,0011,0100,0101,0110,0111,
    1000,1001,1010,1011,1100,1101,1110,1111

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$\binom{n}{k}$	0	1	2	3	4	5	6	7	8
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1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5									
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8									

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    0000,0001,0010,0011,0100,0101,0110,0111,
    1000,1001,1010,1011,1100,1101,1110,1111
  - Legnth-5 sequences:
    00000,00001,00010,00011,
    00100,00101,00110,00111,
    01000,01001,01010,01011,
    01100,01101,01110,01111,
    10000,10001,10010,10011,
    10100,10101,10110,10111,
    11000,11001,11010,11011,
    11100,11101,11110,11111

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0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6									
7									
8									

Pascal's Triangles!

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1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's Triangles!

Build-up Counting, cont'd

- Let's try to come up with a formula:

$$\{n - sequence\ with\ k\ 1's\} =$$
$$= 0 \bullet \{(n-1) - sequence\ with\ k\ 1's\} \cup 1 \bullet \{(n-1) - sequence\ with\ (k-1)1's\}$$
$$= \binom{n-1}{k} + \binom{n-1}{k-1}$$

  - Hm, looks like induction!

- Sum rule:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

- Base cases:

$$\binom{1}{0} = 1, \binom{1}{1} = 1$$

  - More generally, for any n:

$$\binom{n}{0} = 1, \binom{n}{n} = 1$$

$\binom{n}{k}$	0	1	2	3	4	5	6	7	8
0	1								
1	1	1							
2	1	2	1						
3	1	3	3	1					
4	1	4	6	4	1				
5	1	5	10	10	5	1			
6	1	6	15	20	15	6	1		
7	1	7	21	35	35	21	7	1	
8	1	8	28	56	70	56	28	8	1

Pascal's Triangles!

Build-up Counting for Goody Bags

- Let $Q(n,k) =$ number of goody-bags of n candies with k colors
- First, tinker!
  - Suppose we have n candies but only 1 color (red)
  $$Q(n,1) = 1$$
  - Suppose we have zero candies and k colors
  $$Q(0,k) = 1$$
  - Suppose we have 1 candy and k colors
  $$Q(1,k) = k$$
- Build-up counting: there are $(n+1)$ types of goody-bag:
  - Goody-bags that contain 0 red candies (and $k-1$ other colors):
  $$Q(n,k-1)$$
  - Goody-bags that contain exactly 1 red candy (so if I remove it, I have none):
  $$Q(n-1,k-1)$$
  - Goody-bags that contain exactly 2 red candies:
  $$Q(n-2,k-1)$$
  ...
  $$Q(0,k-1)$$

Build-up Counting for Goody Bags, cont'd

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- I can express $Q(n,k)$ recursively as follows:

$$Q(n,k) = Q(n,k-1) + Q(n-1,k-1) + \cdots + Q(0,k-1)$$

- Let's look at the recursive table

k (columns), n (rows)

$Q(n,k)$	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1
1	1										
2	1										
3	1										
4	1										
5	1										

Build-up Counting for Goody Bags, cont'd

Rensselaer

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0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3								
2	1	3	6								
3	1	4	10								
4	1	5	15								
5	1	6	21								

Build-up Counting for Goody Bags, cont'd

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- I can express $Q(n,k)$ recursively as follows:

$$Q(n,k) = Q(n,k-1) + Q(n-1,k-1) + \cdots + Q(0,k-1)$$

- Let's look at the recursive table

n \ k: $Q(n,k)$	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	1001
5	1	6	21	56	126	252	462	792	1287	2002	3003

- What's another way to group $Q(n,k)$?
  - All goody-bags that contain at least 1 red candy (already have a k colors) :

$$Q(n-1,k)$$

  - Plus all bags that have no red candies (have at most $k-1$ colors):

$$Q(n,k-1)$$

Build-up Counting for Goody Bags, cont'd

- I can express $Q(n,k)$ recursively as follows:
$$Q(n,k) = Q(n,k-1) + Q(n-1,k-1) + \cdots + Q(0,k-1)$$
- Let's look at the recursive table

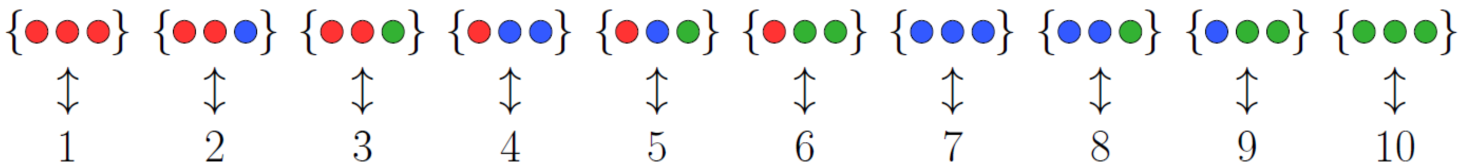
k (columns), n (rows)

$Q(n,k)$	1	2	3	4	5	6	7	8	9	10	11
0	1	1	1	1	1	1	1	1	1	1	1
1	1	2	3	4	5	6	7	8	9	10	11
2	1	3	6	10	15	21	28	36	45	55	66
3	1	4	10	20	35	56	84	120	165	220	286
4	1	5	15	35	70	126	210	330	495	715	1001
5	1	6	21	56	126	252	462	792	1287	2002	3003

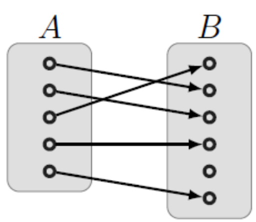
- **Challenge problems we had earlier.**
  - (5 candies, 10 colors) → 2002 goody-bags.
  - How many 5 goody-bag bulk packs (goody-bags have 3 candies of 3 colors)?
    - There are 10 types of goody-bag; 5 in a bulk pack. So we need
$$Q(5,10) = 2002$$

Counting One Set By Counting Another: Bijection

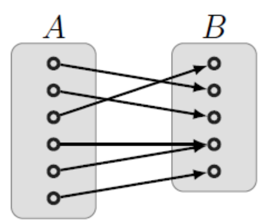
- We saw that there are 10 goody-bags with 3 candies of 3 colors
  - Can label those goody bags using $\{1,2,\ldots,10\}$



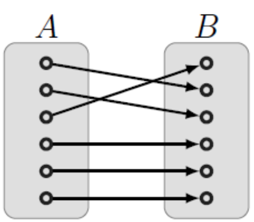
- There is a 1-1 correspondence between goody-bags and the set $\{1,2,\ldots,10\}$
  - We call this a *bijection*!
- Some examples of bijections and other relations



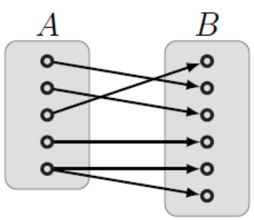
1-1, but **not** onto.
(injection, $A \xrightarrow{INJ} B$)
$|A| \le |B|$



onto; **not** 1-1
(surjection, $A \xrightarrow{SUR} B$)
$|A| \ge |B|$



onto **and** 1-1
(bijection, $A \xrightarrow{BIJ} B$)
$|A| = |B|$



not a function

Counting One Set By Counting Another: Bijection, cont'd

- $A \xrightarrow{BIJ} B$ implies $|A| = |B|$. Can count A by counting B
- Count menus by counting sequences $\{BLD\}$. Works because
  - Every sequence specifies a distinct menu (1-to-1 mapping).
  - Every menu corresponds to a sequence (the mapping is onto).

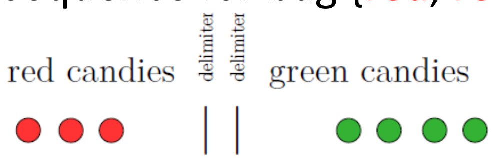
Goody Bags Using Bijection to Binary Sequences

- Suppose we have 3 candy colors: red, green, blue
- Consider the 7-candy goody-bag: {red, red, blue, blue, blue, green, green}

  - Order all candies according to color: red first, then blue, then green
  - Color doesn't matter anymore
  - Hm, this looks like a binary sequence:

$$001000100$$

  - What is the binary sequence for bag {red, red, red, green, green, green, green}?



$$000110000$$

    - Can represent all goody bags with 9 bits

Goody Bags Using Bijection to Binary Sequences, cont'd

- **More examples.**

$$00100010101000 \rightarrow \circ\circ|\circ\circ\circ|\circ|\circ|\circ\circ\circ \leftrightarrow \{2●, 3●, 1●, 1●, 3●\}$$

$$1000011010000 \rightarrow |\circ\circ\circ\circ||\circ|\circ\circ\circ\circ \leftrightarrow \{0●, 4●, 0●, 1●, 4●\}$$

- In general, if we have n candies and k colors, how many delimiters do we have?

$$(k-1)$$

  – i.e., number of goody-bags with n candies of k colors =

  number of $(n+k-1)$-bit sequences with $(k-1)$ 1's

$$Q(n,k) = \binom{n+k-1}{k-1}$$

- This is called **sampling with replacement**
- Hm, $\binom{n}{k}$ keeps popping up but we don't have a formula for it.

Permutations and Combinations

- Consider the set $S = \{1,2,3,4\}$
  - All 2-orderings of S are: $\{12,13,14,21,23,24,31,32,34,41,42,43\}$
    - Permutations: order matters
  - All 2-subsets of S are: $\{12,13,14,23,24,34\}$
    - Combinations: order doesn't matter
- With n elements, by the product rule, the number of k-orderings is

$$n \times (n-1) \times (n-2) \times \cdots \times \big(n-(k-1)\big) = \frac{n!}{(n-k)!}$$

  - e.g. number of top-3 finishes in 10-person race is

$$10 \times 9 \times 8 = \frac{10!}{7!}$$

Permutations and Combinations, cont'd

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- Consider the set $S = \{1,2,3,4\}$
  - All 2-orderings of S are: $\{12,13,14,21,23,24,31,32,34,41,42,43\}$
    - Permutations: order matters
  - All 2-subsets of S are: $\{12,13,14,23,24,34\}$
    - Combinations: order doesn't matter
- Here's another way to count all k-orderings
  - First, pick a k-subset, then re-order it in all possible ways
  - How many k-subsets are there?

$$\binom{n}{k}$$

  - How many ways can we re-order a set?

$$k \times (k-1) \times \cdots \times 1 = k!$$

number of k-orderings $=$ number of k-subsets $\times\, k!$ [**product rule**]

$= \binom{n}{k} \times k!$ [**bijection to sequences with *k* 1's**]

Permutations and Combinations, cont'd

- First method

$$n \times (n-1) \times (n-2) \times \cdots \times \big(n-(k-1)\big) =$$

$$= \frac{n!}{(n-k)!}$$

- Second method

number of k-orderings $=$ number of k-subsets $\times\, k!$ **[product rule]**

$$\frac{n!}{(n-k)!} = \binom{n}{k} \times k!$$ **[bijection to sequences with k 1's]**

- Finally,

number of k-subsets $= \binom{n}{k} = \frac{n!}{(n-k)!k!}$

- **Exercise.** How many 10-bit binary sequences are there with four 1's?

Binomial Theorem: $(x+y)^n = \sum_{i=1}^{n} \binom{n}{i} x^i \, y^{n-1}$

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- We learn the formulas for small n in high school, e.g.,:

$$(x+y)^3 = xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

  - (All length-3 binary sequences $b_1b_2b_3$ where each $b_i \in \{x, y\}$)

- Of course, as we increase n the number of terms grows quickly, so we want a nice clean formula
  - The Binomial Theorem!

- In general, each term has combined power n:

$$(x+y)^n = x^n + (?)x^{n-1}y + (?)x^{n-2}y^2 + \cdots + (?)xy^{n-1} + y^n$$

  - How many strings with $(n-1)$ x's (first coefficient)?

$$\binom{n}{n-1}$$

  - How many strings with $(n-2)$ x's (second coefficient)?

$$\binom{n}{n-2}$$

- Finally,

$$(x+y)^n = x^n + \binom{n}{n-1} x^{n-1}y + \binom{n}{n-2} x^{n-2}y^2 + \cdots + \binom{n}{1} xy^{n-1} + y^n$$

Binomial Theorem Example

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- What is the coefficient of x^7 in the expansion of $\left(\sqrt{x} + 2x\right)^{10}$
  - Need $\left(\sqrt{x}\right)^i (2x)^{10-i} \sim x^7$, which implies $i = 6$
  - The x^7 term is $\binom{10}{6}\left(\sqrt{x}\right)^6 (2x)^4$
  - Coefficient of x^7 is $\binom{10}{6} \times 2^4 = 3360$

General Approach to Counting Complex Objects

- To count complex objects, give a sequence of “instructions” that can be used to construct a complex object.
  - *Every* sequence of instructions gives a *unique* complex object.
  - There is a sequence of instructions for *every* complex object.
- Count the number of possible *sequences* of instructions, which equals the number of complex objects.