MIDTERM: 90 Minutes

Last Name: Solutions

First Name: ____________

RIN: ____________

Section: ____________

Answer **ALL** questions. You may use one double sided $8\frac{1}{2} \times 11$ crib sheet.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.
You **MUST** show **CORRECT** work, **even on multiple choice questions**, to get credit.

GOOD LUCK!

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

1 Circle one answer per question. 10 points for each correct answer.

(1) Compute $S = \sum_{i=1}^{2}\sum_{j=1}^{4}(i+j)$.

$\sum_{i=1}^{2}\sum_{j=1}^{4} i + \sum_{i=1}^{2}\sum_{j=1}^{4} j$

C

No Work No Credit

[A] $S = 28$.

[B] $S = 30$.

[C] $S = 32$.

[D] $S = 34$.

[E] None of the above.

$= 4\sum_{i=1}^{2} i + \sum_{i=1}^{2}\frac{4\cdot 5}{2} = 4\cdot\frac{2(3)}{2} + 2\cdot\frac{4\cdot 5}{2}$

$= 12 + 20 = \boxed{32}$

(2) Estimate. Which approximation is closest to $S = \sum_{i=1}^{2^{50}} i$.

$n = 2^{50}$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{2^{50}(2^{50}+1)}{2}$

$\approx \frac{2^{100}}{2} = \frac{(2^{10})^{10}}{2} \approx \left(\frac{10^3}{2}\right)^{10} = \boxed{\frac{1}{2}\times 10^{30}}$

D

No work No credit

[A] $S \approx 1275$.

[B] $S \approx 10^{10}$.

[C] $S \approx 10^{20}$.

[D] $S \approx 10^{30}$.

[E] $S \approx 10^{40}$.

(3) What is the correct asymptotic behavior (order analysis) for the function $S(n) = n\sqrt{n}$.

$n\sqrt{n} = n^{3/2}$

E

[A] $S(n) \in \Theta(n)$. ✗

[B] $S(n) \in \Theta(n^2)$. ✗

[C] $S(n) \in \Theta(n^3)$. ✗

[D] $S(n) \in \Theta(n^4)$. ✗

[E] None of the above.

Polynomial order must match.

(4) What is the correct asymptotic behavior (order analysis) for the sum $S(n) = \sum_{i=0}^{n^2} 2^i$.

C

No Work No credit

[A] $S(n) \in \Theta(2^n)$.

[B] $S(n) \in \Theta((2^n)^2)$.

[C] $S(n) \in \Theta(2^{n^2})$.

[D] $S(n) \in \Theta(2^{2n^2})$.

[E] None of the above.

$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$

$\sum_{i=0}^{n^2} 2^i = 2^{n^2+1} - 1$

$= 2\cdot 2^{n^2} - 1 \in \boxed{\theta(2^{n^2})}$

(5) Compute gcd(1045, 2310). That is, compute the greatest common divisor of 1045 and 2310.

$\gcd(m,n) = \gcd(\text{rem}(n,m), m)$

C

No Work, No credit.

[A] 5.

[B] 10.

[C] 55.

[D] 95.

[E] None of the above.

$2\times 1045 = 2090 \quad 2310 = 2\times 1045 + 220 \quad 1045 = 4\cdot 220 + 165$

$\gcd(1045, 2310) = \gcd(220, 1045)$

$= \gcd(165, 220)$

$= \gcd(55, 165)$

$= \gcd(0, 55)$

$= \boxed{55}$

(6) Which degree sequence could be the degree sequence of a friendship network (simple graph)?

E

No Work No Credit

[A] [5, 3, 3, 2, 1]. ✗ 5 vertices → max degree = 4

[B] [3, 3, 2, 1]. × Sum of degrees = 9 = 2E contradiction

[C] [3, 3, 3, 3, 3]. ×

[D] [4, 4, 3, 2, 1]. × Degree 4 vertices connected to all others → min degree ≥ 2

[E] None of the above.

(7) You wish to color the graph on the right so that linked vertices do not get the same color. What is the minimum number of colors needed (the chromatic number).

C

[A] 2

[B] 3

[C] 4

[D] 5

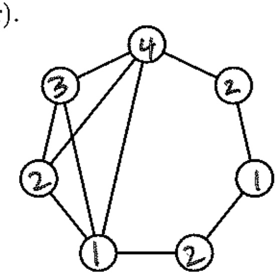
[E] 6

Can do with 4 colors. →

4 are needed for the 4-clique

↓

4 is min.



(8) Boys X, Y, Z and girls A, B, C have the preference lists shown. Which of these is a stable matching?

A

[A] A–X, B–Y, C–Z.

[B] A–X, B–Z, C–Y.

[C] A–Y, B–X, C–Z.

[D] A–Z, B–Y, C–X.

[E] None of the above or there is no stable matching.

Dating algorithm

XYZ go to A → rejects YZ

YZ go to B → rejects Z

Z goes to C → A–X B–Y C–Z

	X	Y	Z
1.	A	A	A
2.	B	B	B
3.	C	C	C

	A	B	C
1.	X	X	X
2.	Y	Y	Y
3.	Z	Z	Z

(9) What is the minimum number of children that guarantees at least two have the same first and last initial?

D.

[A] 26

[B] 26 + 1

[C] 26^2

[D] $26^2 + 1$

[E] None of the above.

First → 26 ways

Second → 26 ways

First × Second → 26^2 ways

Pigeonhole for each First × Second

Need more pigeons (students) → $\boxed{26^2+1}$

(10) A race has 6 runners. In how many ways can the gold, silver and bronze medal be given?

C

[A] 6^3

[B] $\binom{6}{3}$

[C] $6 \times 5 \times 4$

[D] 6!

[E] None of the above.

6 for gold

5 for silver given gold

4 for bronze given gold, silver

Product Rule: $6 \times 5 \times 4$

(11) A shirt matches 2 pants. My blue tie matches 3 shirts. My red tie matches 4 shirts. How many matching outfits can I wear? (In a matching outfit, the shirt must match the tie and pants.)

B

- A 9.
- B 14.
- C 18.
- D 24.
- E None of the above.

outfits: blue tie $\leq$ 3 shirts $\leq$ 2 pants $\to 3\times 2$

red tie $\leq$ 4 shirts $\leq$ 2 pants $\to 4\times 2$

Sum $7\times 2 = \boxed{14}$

(12) A shelf has some books. Alice picks 3 to read and Bob picks 4 to read. Which must be true?

E

- A Alice has more ways to pick her books than Bob has for picking his books. ×
- B Bob has more ways to pick his books than Alice has for picking her books. ×
- C Alice and Bob have the same number of ways for picking their books. ×
- D Alice and Bob cannot have the same number of ways for picking their books. ×
- E None of the above.

n = # books.

Alice $\binom{n}{3}$ Bob $\binom{n}{4}$

$n \leq 7$ Alice > Bob

$n > 7$ Alice < Bob

$n = 7$ Alice = Bob

(13) How many subsets of $\{1, 2, 3, 4, 5\}$ contain 1 and 2 or contain 3 and 4?

C

- A 10.
- B 12.
- C 14.
- D 16.
- E None of the above.

Contain 1,2 : 2^3 subsets.

Contain 3,4 : 2^3 subsets.

Contain 1,2 and 3,4 : 2.

Inclusion-Exclusion: $2^3+2^3-2 = 16-2 = \boxed{14}$

(14) How many different shortest paths from A to B are there?

E

No work No credit.

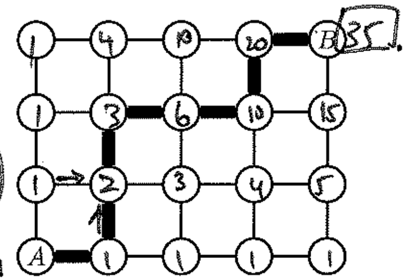
- A 28
- B 32
- C 34
- D 36
- E None of the above.

7 steps

4 are right steps, 3 are up

Choose 4 right from 7 = $\binom{7}{4}$

$\binom{7}{4} = \frac{7\times \cancel{6}\times 5\times \cancel{4}}{1\times \cancel{2}\times \cancel{3}\times \cancel{4}} = 7\times 5 = 35$



Build up $N(x,y) = N(x-1,y) + N(x,y-1)$

(15) You write all 1,000 numbers in $\{0, 1, \ldots, 999\}$. How many times did you write the digit 1?

B

No work No credit.

- A 275.
- B 300.
- C 325.
- D 350.
- E None of the above.

Write 000, 001, ... , 999 ← 1000 numbers.

digits written = 3×1000

Each digit written same # of times.

$\therefore$ Each is written $\frac{3\times 1000}{10}$ times

$= \boxed{300}$

2 **Prove or disprove: the integer x is odd if and only if $x^2 - 1$ is divisible by 8.**

If and only if means prove two implications! ← 50%.

Prove $8 | x^2-1 \rightarrow x$ is odd.

Suppose $8|x^2-1$

$\rightarrow x^2-1$ is even

$\rightarrow x^2$ is odd

$\rightarrow x$ is odd (proved in class). ∎ ← 80% for generally correct proof of one direction.

Prove x is odd $\rightarrow 8 | x^2-1$

Let $x = 2k+1$

x is odd $\rightarrow x^2-1 = (x-1)(x+1)$

= product of consecutive even numbers.

$= (2k)(2k+2) = 4k(k+1)$

if k is even then $4k$ is div by 8

if k is odd then $4(k+1)$ is div by 8

In both cases x^2-1 is div by 8 ∎ ← 100%.

Grading

50%: 2 implications ~~proved~~ recognized

80%: 1 implication generally correct

100%: Both implications generally correct

3 **For $n \in \mathbb{N}$, prove that** $\text{REMAINDER}(n, 9) = \text{REMAINDER}(\text{sum of } n\text{'s digits}, 9)$.

E.g. 725 has remainder 5 modulo 9. The sum of the digits is 14, which also has remainder 5 modulo 9.

Basic Knowledge. $a \equiv b$ $\quad c \equiv d$ $\quad (\text{mod } d)$

$\rightarrow$ $a + c \equiv b + d$ ①

$ac \equiv bd$. ②

$a^i \equiv b^i$ ③

$(\text{mod } 9)$

$10^0 \equiv 1$

$10 \equiv 1$

$10^i \equiv 1$

50%

Let n have digits $a_0 a_1 a_2 \ldots a_K$

80%

$\therefore$ $n = \sum_{i=0}^{K} a_i 10^i$

$10^i \equiv 1 \quad (\text{mod } 9)$

$a_i \equiv a_i \quad (\text{mod } 9)$

$\rightarrow$ $a_i 10^i \equiv a_i \quad (\text{mod } 9)$

100%.

By property ① $\underbrace{\sum_i a_i 10^i}_{n} \equiv \sum_i a_i \quad (\text{mod } 9)$

$$\boxed{n \equiv \sum_i a_i \quad (\text{mod } 9.}$$

Grading

50%: Show some general useful knowledge + tinkering.

80%: Some progress

100%: Generally Correct

4 **Prove by induction for all $n \geq 1$:**

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{(n-1)}{n!}+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}.$$

$$P(n): \sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$$

Tinker: $P(1)$: $\frac{1}{2!} = 1 - \frac{1}{2!}$ ✓

$P(2)$: $\frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6} = 1 - \frac{1}{6} = 1 - \frac{1}{3!}$ ✓

Proof By Induction

Base Case: $P(1)$ is true.

Induction Step: Assume $P(n)$: $\sum_{i=1}^{n} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+1)!}$

Prove $P(n+1)$: $\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = 1 - \frac{1}{(n+2)!}$ 50%.

$$\sum_{i=1}^{n+1} \frac{i}{(i+1)!} = \sum_{i=1}^{n} \frac{i}{(i+1)!} + \frac{n+1}{(n+2)!}$$ 80%.

$$= 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \quad \text{(induction hypothesis.}$$

$$= 1 - \frac{1}{(n+2)!}\Big[\underbrace{n+2-(n+1)}_{1}\Big]$$

$$= 1 - \frac{1}{(n+2)!} \quad \text{as was to be shown.}$$ 100%.

By induction $P(n)$ is true for all $n \geq 1$

Grading

50%: Tinkering, Understood Problem, Setup Basic Induction infrastructure.

80%: Some progress, e.g. link $P(n+1)$ to $P(n)$.

100%: Basically correct induction.

5 A_n **satisfies the recurrence below. Find a formula for A_n and prove your answer.**

$$A_0 = 1, A_1 = 2 \text{ and } A_n = A_{n-1} + 2A_{n-2} \text{ for } n \geq 2$$

$\boxed{\text{TINKER}}$

n	0	1	2	3	4	5	6	...
A_n	1	2	4	8	16	32	64	...

50%

Guess $A_n = 2^n$ $\quad n \geq 0$.

$P(n)$: $A_n = 2^n$

Proof by induction (Strong).

Base Case: $A_0 = 1 = 2^0$ ✓
$A_1 = 2 = 2^1$ ✓

} $\cdot P(1)$ and $P(2)$ are true.

NEED 2 base cases

Induction Step:

Assume: $A_0 = 2^0, A_1 = 2^1, \ldots, A_n = 2^n \quad n \geq 1$

Prove: $A_{n+1} = 2^{n+1}$

80%

By the recursion

$$\begin{aligned} A_{n+1} &= A_n + 2 \cdot A_{n-1} \\ &= 2^n + 2 \cdot 2^{n-1} \\ &= 2^n + 2^n \\ &= 2 \times 2^n \\ &= 2^{n+1} \quad \text{as was to be shown} \end{aligned}$$

100%.

By induction, $A_n = 2^n$ for all $n \geq 0$.

Grading

50%: Tinker and make right guess

80%: Proof by strong induction framework.

100%: Basically correct induction and linking A_{n+1} to A_n, A_{n-1}

6 How many subsets of $\{1,2,3,4,5,6\}$ contain some consecutive numbers?.

For example the subset $\{1,2,4,5\}$ has consecutive numbers but the subset $\{1,3\}$ does not.

80% if you list all $2^6 = 64$ subsets and count which have consec #s.

TINKER For $\{1\} \to 0$ subsets
For $\{1,2\} \to 1$ subset

50%

Build Up Counting.

Let $W(n)$ be the # of subsets with some consecutive #s when the base set has n consec #s.

Let $Q(n)$ be the # of subsets with $\boxed{\text{NO}}$ consecutive #s when the base set has n consec #s.

80%

$W(n) = 2^n - Q(n)$

Subsets in $Q(n)$

$\{1,2,3,4,5,6,\dots,n\}$

→ Contain 1 → No 2 → $\underbrace{\{1 + \text{subset of } 3\dots n\}}_{\text{\# of such subsets is } Q(n-2)}$

→ Do not contain 1 → $\underbrace{\{\text{subset of } 2,3,\dots n\}}_{\text{\# of such subsets is } Q(n-1)}$.

$\therefore\ Q(n) = Q(n-1) + Q(n-2)$.

$Q(0) = 1$ $\quad \phi$

$Q(1) = 2$ $\quad \phi, \{1\}$

$Q(2) = 3.$ $\quad \phi, \{1\}, \{1,2\}$.

100%

n	0	1	2	3	4	5	6
$Q(n)$	1	2	3	5	8	13	21

~~$W(n) = 64 - 34$~~

$W(n) = 2^6 - Q(6)$

$= 64 - 21$

$= 43$

ANSWER

Grading

50%: Tinker and understand problem

80%: Formulate appropriate notation

100%: Basically correct recursion and answer.