Midterm

110 Minutes

First Name: Solutions

Last Name: ____________

RIN: ____________

NO COLLABORATION or electronic devices.
Any violations will result in an F.
No questions allowed during the test unless you think there is a mistake.

GOOD LUCK!

10 points per correct multiple-choice answer. Circle exactly one answer.
20 points per correct answer to Problems 2-6.
You **MUST** show **CORRECT** work to get credit.
Correct answers with no explanation will get a 0.

1	2	3	4	5	6	Total
150	20	20	20	20	20	250

1. What is the asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n^2} i^2$? [common sum]

A $\Theta(n^3)$
B $O(n^4)$
C $\Theta(n^5)$
D $O(n^6)$
E None of the above.

$$S(n) = \frac{n^2(n^2+1)(2n^2+1)}{6} \in O(n^6)$$

2. What is the asymptotic behavior of the function $S(n) = n^{1.5}$?

A $\Theta(n\sqrt{n})$
B $\Theta(n^2)$
C $\Theta(n^2\sqrt{n})$
D $\Theta(n^3)$
E None of the above.

$$n^{1.5} = n\sqrt{n}$$

3. What is the value of the sum $S = \sum_{i=1}^{10}\sum_{j=1}^{5} i$?

A 200
B 225
C 250
D 275
E 300

$$S = \sum_{i=1}^{10} i \sum_{j=1}^{5} 1 = 5\sum_{i=1}^{10} i = 5 \cdot \frac{\cancel{10}^{5} \cdot 11}{\cancel{2}}$$
$$= 275$$

4. What is the correct asymptotic behavior of the sum $S(n) = \sum_{i=1}^{n} i^{10}$?

A $\Theta(n^{10})$
B $\Theta(n^{11})$
C $\Theta(n^{12})$
D $\Theta(n^{13})$
E None of the above.

Integration method: $\int_1^{n+1} x^{10}dx = \frac{x^{11}}{11}\Big|_1^{n+1}$

$\in \Theta(n^{11})$

SHOW WORK

5. We know that $gcd(290, 310) = 290x + 310y$. Which of the following values are possible for x and y?

A $x = 4, y = -10$
B $x = 8, y = -15$
C $x = 10, y = -20$
D $x = 14, y = -15$
E None of the above.

$gcd(290, 310) = 10 \times gcd(29, 31)$
$gcd(29, 31) = gcd(2, 29)$ [$2 = 31 - 29$]
$gcd(2, 29) = gcd(1, 2)$ [$1 = 29 - 14 \cdot 2$
$= 29 - 14 \cdot 31 + 14 \cdot 29$
$= 15 \cdot 29 - 14 \cdot 31$]

SHOW WORK

6. What is the remainder when 10^{100} is divided by 11?

   A -2

   B -1

   C 0

   D 1

   E 2

$10 \equiv -1 \pmod{11}$

$10^{100} \equiv (-1)^{100} \pmod{11}$

SHOW WORK

7. Consider a graph G with degree sequence $[2,2,2,2,2]$. How many edges does G have?

   A 4

   B 6

   C 8

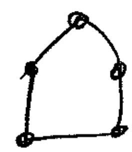
   D 10

   E None of the above.

Handshaking Thm:

$2\#\text{ edges} = \sum_i \delta_i = 10$

$\#\text{ edges} = 5$



SHOW WORK

8. Consider a graph G with degree sequence $[3,2,2,2,2]$. How many edges does G have?

   A 8

   B 9

   C 10

   D 11

   E None of the above.

Sum of degrees is odd.
Graph doesn't exist.

9. There are 10 types of people in the world: those who know binary and those who don't. What is 2^{11}?

   A 128 2^7

   B 256 2^8

   C 512 2^9

   D 1024 2^{10}

   E None of the above.

$2^{11} = 2048$

10. Suppose FOCS has 6 sections (with 33 students per section). Each student shakes hands only with students who are in different sections. What do we know?

    A The number of students who shake hands with an odd number of students is even.

    B The number of students who shake hands with an odd number of students is odd.

    C There are 198×165 handshakes in total.

    D The number of handshakes cannot be determined.

    E None of the above.

odd people is even!

SHOW WORK

11. How many subsets of $\{1, 2, 3, 4, 5, 6\}$ contain at least one even number?

- A 2^3
- B 2^4
- C $2^6 - 2^3$
- D $2^5 - 8$
- E None of the above.

Total # subsets is 2^6

subsets w/ no even # is 2^3 (subsets of $\{1, 3, 5\}$)

subsets w/ at least one even number is $2^6 - 2^3$.

12. Consider all 7-bit binary strings with a 1 in the first position and a 0 in the last position? How many such strings are there?

- A 16
- B 32
- C 64
- D 128
- E None of the above.

$1\, b_1 b_2 b_3 b_4 b_5\, 0$

Essentially, all 5-bit strings

13. Suppose you guess randomly on all 15 multiple-choice questions and you answer 5 correctly. What is the number of all 5-question sets (e.g., a 5-question set is $\{q_1, q_4, q_7, q_{11}, q_{12}\}$)?

- A $\binom{15}{5}$
- B $\binom{10}{5}$
- C $15 \times 14 \times 13 \times 12 \times 11$
- D 15^5
- E None of the above.

$n = 15, \quad k = 5$

14. In how many ways can you misspell WINTER, assuming you use the same letters?

- A 6!
- B 2^6
- C $2^6 - 1$
- D $6! - 1$
- E None of the above.

All permutations, minus the correct spelling

SHOW WORK

15. What is the last digit of 11^5?

- A 0
- B 1
- C 2
- D 3
- E None of the above.

$11 \equiv 1 \pmod{10}$

$11^5 \equiv 1^5 \pmod{10}$

Problem 2. Prove that $\sqrt{12}$ is irrational.

Method 1:

Proof by contradiction. progress (50%)

suppose $\sqrt{12}$ is rational.

Let p_*, q_* be the smallest pair s.t. $\frac{p_*}{q_*} = \sqrt{12}$.

(no common factors). 80%

Then $p_*^2 = q_*^2\, 3 \cdot 4$.

In HW problem, we proved that p_* must be a multiple of 3, 100%

hence q_* must be a multiple of 3. $(p_* = 2 \cdot 3 \cdot k)$

Method 2:

In homework, we proved $\sqrt{3}$ is irrational.

Suppose for a contradiction that $2\sqrt{3}$ is rational, i.e.,

$2\sqrt{3} = \frac{p^*}{q^*}$. Then $\sqrt{3} = \frac{p^*}{2q^*}$, i.e.

$\sqrt{3}$ is rational. Contradiction.

Problem 3. What is the last digit of 102^{1211}? [Hint: $11^2 = 121$]

$$102 \equiv 2 \pmod{10}$$

$$102^{1211} \equiv 2^{1211} \pmod{10}$$

50 %

$$1211 = 1 + 10 \cdot 11^2.$$

$$2^9 = 512 \equiv 2 \pmod{10}$$

$$2^{10} = 1024 \equiv 4 \pmod{10}$$

$$2^{11} = 2048 \equiv 8 \pmod{10}$$

80 %

$$\begin{aligned} 2^{1211} = 2 \cdot \left(\left(2^{11}\right)^{11}\right)^{10} &\equiv 2 \cdot \left(\left(2^{3}\right)^{11}\right)^{10} \pmod{10} \\ &\equiv 2 \cdot \left(\left(2^{11}\right)^{3}\right)^{10} \\ &\equiv 2 \cdot \left(\left(2^{3}\right)^{3}\right)^{10} \\ &\equiv 2 \cdot \left(2^{9}\right)^{10} \\ &\equiv 2 \cdot 2^{10} \\ &\equiv 2 \cdot 4 \\ &\equiv 8 \pmod{10} \end{aligned}$$

100 %

Problem 4. Prove using induction that $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n\cdot(n+1)}=\frac{n}{n+1}, \forall n\geq 1.$

Base case: $\frac{1}{1\cdot 2}=\frac{1}{1+1}$. ✓

50%

Induction Step: Assume true for n.

$P(n)$: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.

80%

Need to show $P(n+1)$: $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n+1)(n+2)}=\frac{n+1}{n+2}$.

$$\underbrace{\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}}_{\text{[induction hypothesis]}}+\frac{1}{(n+1)(n+2)}=$$

$$=\frac{n}{n+1}+\frac{1}{(n+1)(n+2)}$$

100%

$$=\frac{n(n+2)+1}{(n+1)(n+2)}$$

$$=\frac{n^2+2n+1}{(n+1)(n+2)}=\frac{(n+1)^{2}}{\cancel{(n+1)}(n+2)}=\frac{n+1}{n+2}.\ ✓$$

Problem 5. Consider the recurrence $A_0 = 0, A_1 = 1, A_n = 2A_{n-1} - A_{n-2}$. Guess a formula for A_n and prove it using induction. Tinker, tinker, tinker.

$A_0 = 0$

$A_1 = 1$

$A_2 = 2 \cdot 1 - 0 = 2$

$A_3 = 2 \cdot 2 - 1 = 3$

$A_4 = 2 \cdot 3 - 2 = 4$

$A_5 = 2 \cdot 4 - 3 = 5$

50 %

Guess: $A_n = n$.

Base cases: $A_0 = 0$

$A_1 = 1$

80 %

Induction step: Assume true for n.

Need to show $A_{n+1} = n+1$.

$$A_{n+1} = 2 \cdot A_n - A_{n-1}$$

$$= 2 \cdot n - (n-1)$$

$$= n+1. \quad \checkmark$$

100 %

Problem 6. For a graph G, the complement $\overline{G}$ has the same vertices, but the edges in $\overline{G}$ are the complement of the edges in G: distinct vertices u and v are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

Prove that if G is regular, then $\overline{G}$ is regular. [In a regular graph, all vertices have the same degree.]

Suppose all nodes in G have degree δ. Let the number of nodes be n. | 50 %

Then each vertex in $\overline{G}$ has exactly $n-\delta-1$ neighbors | 80 %

(all the vertices which are <u>not</u> neighbors in G). | 100 %

Scratch

Scratch