FINAL: 180 Minutes

Last Name: ______________________

First Name: ______________________

RIN: ______________________

Section: ______________________

Answer **ALL** questions. You may use **two** double sided $8\frac{1}{2} \times 11$ crib sheets.
You **MUST** show **CORRECT** work (even for multiple choice) to receive full credit.
NO COLLABORATION or electronic devices. Any violations result in an F.
NO questions allowed during the test. Interpret and do the best you can.

GOOD LUCK!

1	2	3	4	5	6	Total
200	30	30	30	30	30	350

1 Circle at most one answer per question. 10 points for each correct answer.

(1) n is a natural number and C is a real. True or false: $\exists C > 0 : \forall n \geq 1 : 10n^2 \geq C(n^3 + n)$.

$\boxed{\text{A}}$ True.

$\boxed{\text{B}}$ False.

$\boxed{\text{C}}$ It depends on C.

$\boxed{\text{D}}$ It depends on n.

$\boxed{\text{E}}$ None of the above.

(2) n is a natural number and C is a real. True or false: $\forall n \geq 1 : \exists C > 0 : 10n^2 \geq C(n^3 + n)$.

$\boxed{\text{A}}$ True.

$\boxed{\text{B}}$ False.

$\boxed{\text{C}}$ It depends on C.

$\boxed{\text{D}}$ It depends on n.

$\boxed{\text{E}}$ None of the above.

(3) What is the set $\mathcal{A}$, recursively defined on the right.

$\boxed{\text{A}}$ $\mathbb{N}$.

$\boxed{\text{B}}$ $\mathbb{Z}$.

$\boxed{\text{C}}$ $\mathbb{Q}$.

$\boxed{\text{D}}$ $\mathbb{R}$.

$\boxed{\text{E}}$ None of the above.

(1) $1 \in \mathcal{A}$.
(2) $x, y \in \mathcal{A} \to x + y \in \mathcal{A}$ AND $x - y \in \mathcal{A}$.
(3) Nothing else is in $\mathcal{A}$.

(4) $T_1 = 1$ and $T_n = T_{n-1} + \sqrt{n}$ for $n > 1$. Estimate T_{100}?

$\boxed{\text{A}}$ 7.

$\boxed{\text{B}}$ 70.

$\boxed{\text{C}}$ 700.

$\boxed{\text{D}}$ 7000.

$\boxed{\text{E}}$ 70000.

(5) Compute the sum $S = \sum_{i=1}^{4} \sum_{j=1}^{4} ij^2$.

$\boxed{\text{A}}$ 290.

$\boxed{\text{B}}$ 300.

$\boxed{\text{C}}$ 310.

$\boxed{\text{D}}$ 320.

$\boxed{\text{E}}$ None of the above.

(6) The sum $S(n) = \sum_{i=1}^{n} (i^2 + i)$. Which is true?

$\boxed{\text{A}}$ $S(n) \in \Theta(n)$.

$\boxed{\text{B}}$ $S(n) \in \Theta(n^2)$.

$\boxed{\text{C}}$ $S(n) \in \Theta(n^2 \log n)$.

$\boxed{\text{D}}$ $S(n) \in \Theta(n^3)$.

$\boxed{\text{E}}$ None of the above.

(7) The hour hand on a clock points to 1 o'clock. After 2200^{2200} hours, where will the hour hand be pointing?

$\boxed{\text{A}}$ 2 o'clock.

$\boxed{\text{B}}$ 5 o'clock.

$\boxed{\text{C}}$ 8 o'clock.

$\boxed{\text{D}}$ 11 o'clock.

$\boxed{\text{E}}$ None of the above.

(8) In a graph, the only two vertices with odd degree are u and v. Must there be a path from u to v?

$\boxed{\text{A}}$ Yes, always.

$\boxed{\text{B}}$ No, never.

$\boxed{\text{C}}$ It is possible or not, depending on the number of edges.

$\boxed{\text{D}}$ It is possible or not, depending on the number of vertices.

$\boxed{\text{E}}$ Such a graph cannot exist.

(9) A class has 10 students. How many different debate teams with 5 kids are possible?

$\boxed{\text{A}}$ 10^5.

$\boxed{\text{B}}$ 5^{10}.

$\boxed{\text{C}}$ $10 \times 9 \times 8 \times 7 \times 6$.

$\boxed{\text{D}}$ $\binom{10}{5}$.

$\boxed{\text{E}}$ None of the above.

(10) In how many ways can you distribute ten \$1 bills among three children aged 1,2,3 so that each child gets an amount of money that is at least their age?

$\boxed{\text{A}}$ 8

$\boxed{\text{B}}$ 9

$\boxed{\text{C}}$ 10

$\boxed{\text{D}}$ 11

$\boxed{\text{E}}$ None of the above.

(11) How many 6-bit strings have 00 as a substring. [*Hint*: Let $Q_n = \#\, n$-bit strings $\ldots$.]

$\boxed{\text{A}}$ 21.

$\boxed{\text{B}}$ 32.

$\boxed{\text{C}}$ 43.

$\boxed{\text{D}}$ 64.

$\boxed{\text{E}}$ None of the above.

(12) Roll 3 dice. What are the chances of exactly 2 ones?

$\boxed{\text{A}}$ 5/72.

$\boxed{\text{B}}$ 6/72.

$\boxed{\text{C}}$ 7/72.

$\boxed{\text{D}}$ 8/72.

$\boxed{\text{E}}$ None of the above.

(13) 60 students are split into FOCS (20 boys, 10 girls) and ALGO (10 boys, 20 girls). A random student is picked and it is a girl. What are the chances this student is in FOCS?

$\boxed{\text{A}}$ 1/3.

$\boxed{\text{B}}$ 1/4.

$\boxed{\text{C}}$ 1/5.

$\boxed{\text{D}}$ 1/6.

$\boxed{\text{E}}$ None of the above.

(14) $\mathbb{E}[\mathbf{X}] = 2, \mathbb{E}[\mathbf{Y}] = 3$. What is $\mathbb{E}[2\mathbf{X} + 3\mathbf{Y}]$?

$\boxed{\text{A}}$ 10.

$\boxed{\text{B}}$ 11.

$\boxed{\text{C}}$ 13.

$\boxed{\text{D}}$ 15.

$\boxed{\text{E}}$ None of the above, or not enough information given.

(15) $\mathbb{E}[\mathbf{X}] = 2, \mathbb{E}[\mathbf{Y}] = 3$. What is $\mathbb{E}[\mathbf{X}^2 + 3\mathbf{Y}]$?

$\boxed{\text{A}}$ 10.

$\boxed{\text{B}}$ 11.

$\boxed{\text{C}}$ 13.

$\boxed{\text{D}}$ 15.

$\boxed{\text{E}}$ None of the above, or not enough information given.

(16) Flip a fair coin until 1 or more heads ***or*** 2 or more tails. What is the expected number of flips?

A 1.5.

B 2.5.

C 3.5

D 4.5.

E None of the above.

(17) Flip a fair coin until 1 or more heads ***and*** 2 or more tails. What is the expected number of flips?

A 1.5.

B 2.5.

C 3.5

D 4.5.

E None of the above.

(18) Toss 10 balls randomly into 10 bins. What is the expected number of bins with exactly 1 ball.

A $9^8/10^8$.

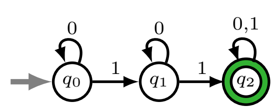
B $9^8/10^9$.

C $9^9/10^8$.

D $9^9/10^9$.

E None of the above.

(19) What is the computing problem solved by the DFA on the right?



A Strings with an even number of 1s.

B Strings with two 1s.

C Strings with at least two 1s.

D Strings with more than two 1s.

E None of the above.

(20) $\mathcal{L}_A$ is reducible to $\mathcal{L}_B$, that is $\mathcal{L}_A \leq_{\mathrm{R}} \mathcal{L}_B$. We know $\mathcal{L}_B$ is decidable. Therefore:

A $\mathcal{L}_A$ must be finite.

B $\mathcal{L}_A$ must be infinite.

C $\mathcal{L}_A$ must be decidable.

D $\mathcal{L}_A$ must be undecidable.

E None of the above.

2 Pick any 11 distinct numbers from $\{1, 2, \ldots, 15\}$. Prove that three are consecutive.

For example, if you pick $\{1, 2, 4, 5, 6, 7, 8, 10, 11, 14, 15\}$ then 4,5,6 are consecutive.

3 $\mathcal{L} = \{111, 11111\}^*$. What strings are in $\mathcal{L}$. Prove your answer.

4 A "Diagonal" binomial sum. Prove by induction: $\displaystyle\sum_{k=0}^{n} \binom{m+k}{k} = \binom{m+n+1}{n}$, for $m, n \geq 0$.

5 Expected number of runs.

A biased coin with probability 1/3 of heads is flipped 10 times. In a run, all consecutive flips are the same, for example HHTHTTTTHH has five runs. Compute the expected number of runs.

6 Give a sketch (high-level pseudocode) for a Turing Machine to Solves $\mathcal{L} = \{0^{\bullet n^2} | n \geq 0\}$.

SCRATCH

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